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Liaison, Schottky Problem and Invariant Theory

Remembering Federico Gaeta

María Emilia Alonso
Enrique Arrondo
Raquel Mallavibarrena
Ignacio Sols
Editors

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Preface

Federico Gaeta (1923–2007) was a Spanish algebraic geometer who was a student of Severi. He is considered to be one of the founders of linkage theory, on which he published several key papers. After many years abroad he came back to Spain in the 1980s. He spent his last period as a professor at Universidad Complutense de Madrid.

In gratitude to him, some of his personal and mathematically close persons during this last station, all of whom benefited in one way or another by his inspiration, have joined to edit this volume to keep his memory alive.

We offer in it surveys and original articles on the three main subjects of Gaeta's interest through his mathematical life.

The volume opens with a personal semblance by Ignacio Sols and a historical presentation by Ciro Ciliberto of Gaeta's Italian period. Then it is divided into three parts, each of them devoted to a specific subject studied by Gaeta and coordinated by one of the editors. For each part, we had the advice of another colleague of Federico linked to that particular subject, who also contributed with a short survey. The first part, coordinated by E. Arrondo with the advice of R.M. Miró-Roig, deals with linkage theory, corresponding to Gaeta's Italian period (see papers G2–G20) and contains, besides a survey, five research articles. The second part was coordinated by R. Mallavibarrena with the advice of J.M. Muñoz-Porras, and is devoted to the Schottky problem, the main interest of Gaeta during his American period; it contains a survey and two research articles (see papers G37–G41). The third part, coordinated by M.E. Alonso with the advice of L. González-Vega, has to do with computation in algebraic geometry, the interest of the elderly years of Federico (see papers G42-G48; G50-G52); here we include, with a short presentation, an unpublished article of Federico Gaeta, as well as a research article inspired by his ideas.

Thanks, then, to those who accepted our invitation to collaborate with research articles to this volume – thanks also to their referees – and thanks to the mentioned authors of the reviews in this volume, including their advice to the editors. And special thanks, finally, to Ciro Ciliberto for his erudite and pleasant recreation of the Italian period of Federico Gaeta.

Part I

Federico Gaeta

Professor Federico Gaeta (1923–2007)

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Federico Gaeta, Among the Last Classics

Ignacio Sols

Abstract. This is a personal approach to the human and academic aspects of the life of Federico Gaeta

Mathematics Subject Classification (2000). Primary 14M06; Secondary 14K25.

Keywords. Linkage, jacobians, Chow forms.

Each society in our modern world, when it is at its best, seeks to become a model of the virtues it celebrates and holds this model up as an example and a goal for young people just beginning their lives and careers. The tragedy of Federico Gaeta was that, at the time of his main achievements, there was insufficient scientific maturity in Spain to appreciate them, and as a result he enjoyed more recognition outside of our frontiers than in his own country.

He had great difficulties in finding an academic position in Spain; once he attained one that was at least marginally suitable, he chose not to settle in but, in view of the political situation, left the country a few years later for a more promising atmosphere. When, decades later, he returned as a weary soldier, the dynamics of Spain had fortunately changed, not only in the political sense, but in the existence of a livelier scientific milieu.

A generation of young Spanish mathematicians working in different new and exciting areas had started from the beginnings of the 1970s to go abroad for updated training in research; at the end of this decade and the beginning of the 1980s, an atmosphere flourished that was more ready to appreciate the achievements of some of our predecessors. In this sense, it is fortunate that Federico Gaeta was, at least at the end, surrounded by the admiration and the affection of young people who had come to know him and his work and looked up to him as an example to follow.

Based on this history, the idea arose of organizing a conference in Madrid in celebration of his retirement; the international participation was splendid and the spirit of cooperation was whole-hearted (I take this occasion to say thanks to the organizers and participants who made it such a magnificent event!). There were, however, no proceedings recorded of this conference, and we felt a lack of something crucial in the historical record to keep the memory of our dear Federico

alive for coming generations as well as for ourselves. The motivation for such a publication has now sadly arisen from the occasion of his passing away. We feel as though we have lost the "last of the living", as I liked to call him, among the mathematicians who thought in the fresh spirit of the intuitive Italians; the great Segre (both of them), Castelnuovo, Enriques, and Severi – his mentor.

He felt far away from "esas cosas cohomolgicas modernas", to which he never thought to have added so much. "Oh, no, no, Sols. Don't say that. Don't compare me with them" he complained when I named him among this golden chain of Italians. But he was evidently pleased. It is not an easy task to draw a true biographical profile of D. Federico Gaeta, since he was in no way a methodical man, and sometimes projected himself and his work as a mess.

When, for instance, he was asked to list his publications in an application for funding for a project, he simply wrote down those that came to his mind at that moment; he seemed unable to systematically carry out such necessary formalities. And this comes through in the notes I took from the conversations I had with him on the occasion of my talk at the conference in Madrid, and later during a most pleasant meeting with Teresa Aréjola, his widow. Nevertheless, I offer them as a token of my appreciation for him and my fortunate relationship with him.

Federico Gaeta was born in Madrid, March 3rd of 1923. He presented his Ph.D. thesis in the same city in 1945, under the guidance of Tomás Rodriguez Bachiller. As noted by D. Federico, mathematicians of that time in our country were people of general, encyclopedic, knowledge – Rodriguez Bachiller himself was a civil engineer. Federico always acknowledged the positive influence of this professor, who provided him with the right books to read, and put him in contact with Francesco Severi to promote his going to Italy with the help of a scholarship.

Federico's Italian period was between 1946 and 1951; he commuted between Madrid and Rome for some years before moving there. He always thought of Severi as having treated him as a son, as one of his family. In fact, the research subject he proposed to Federico was related to that of one of his important acquaintances. He knew that Federico was a friend of Francesco Gherardelli, son of Giuseppe Gherardelli ,who had characterized at that time the complete intersection curves of three-dimensional projective space as those which are arithmetically normal and subcanonical. The research subject proposed by Severi was then the study of the liaison class of the curves "di residuale finito", as the arithmetically normal curves were called at the time. This is how Gaeta came to write, under the advice of Severi, his two famous memoirs on these curves where he characterized them as those linked to complete intersections, the theorem which would be re-proved in modern terms two decades later by Peskine and Szpiro and which is commonly understood as the true starting point of linkage theory.

This was later generalized in the classification of the linkage classes by its Hartshorne-Rao module, and has found generalizations to other notions of linkage. But the spirit of the theory was all in this classical theorem. And, by the way, this is perhaps the right time to say that D. Federico always corrected me or anyone when saying this: "y de Roger Apéry, no lo olvide usted" he requested that we add.

This brings us to the Parisian life of Federico during 1951 and 1952, as an attaché de recherche du CNRS , working with Paul Dubreil, who as is well known was more of an algebraist but who had recently written an article in connection with Gaeta's interest in algebraic geometry: "Sur quelques problèmes concernant les variétés algébriques et la théorie des syzygies des idéeaux de polynomes". This is the time when Federico was trying to get a position in his own country, in a difficult atmosphere that did not give priority to scientific merit, which explains his first failure to become a professor.

But subsequently, in 1952, he got a position as "catedrático" – full professor – in the university of Zaragoza. When I was a student in Zaragoza, the collective memory of Federico Gaeta still remained, partially because he was exceptionally bright but also because of his, let us say, "nonconformism" – he was always involved in some kind of controversy, because he never learned to keep silent when something appeared to him to be wrong. His own explanation was simple, that he was "expedientado" by "rebeldía científica", and also "porque yo era antifranquista" he added. In principle, "expedientado" means suspended from teaching, but it seems he was at least allowed the possibility of going to some other university, which he obviously did not, but he took the occasion of this banishment to leave his country until the political and scientific scene would change.

That is how he showed up in Argentina, in 1957, starting a "periplo" through different American countries. He stayed in Argentina three years in different universities: Buenos Aires, La Plata, San Carlos de Bariloche. He came into contact there with the Spanish geometer Luis Santal, prominent in the field of integral geometry, so Federico also became interested in some applications of algebraic geometry to integral geometry, studying invariant measures on homogeneous spaces.

After that, he went for three years, 1960 to 1963, to Sao Paulo, where he settled – until things changed in Spain – in the University of Buffalo (with nine months in the middle, in 1969, of teaching at the university of Caracas). This is the time when he became interested in the Schottky problem, and the "zugeordnete Form", the name by which he still called the Chow form of projective varieties. At the end of his term in Buffalo, he had a permanent residence and offered his house to Spanish exiles, notable among which was the president of the historic Spanish socialist party – then surviving in the underground – D. Enrique Tierno Galván. His generosity was also extended during the self-imposed scientific exile of Alexander Grothendieck, who lived for a year with Gaeta's family, and whose more genial characteristics the sons of Gaeta still recall with many anecdotes. (For example, when people came to the house to talk with Grothendieck of higher mathematics, he would say to one of the children of Gaeta, "Please, Teresa, play the guitar", and the music was on, and the maths were over).

This willingness to take on problematic situations – perhaps by subscribing to the idea of "al pan, pan, y al vino, vino", freely translated as "never being a diplomat" – was not only a reflection of those special times in Spain but was indicative of Gaeta's character itself, wherever he was. This can be illustrated by a fondly remembered anecdote. In the middle 1970s, I had just recently heard about

Gaeta, as had every young mathematician or student of mathematics in Spain. When being introduced to William F. Lawvere, then a professor at the University of Buffalo, I asked him (how could I have resisted) about Gaeta. "Oh, someone who is always involved in troubles?" he replied. "Oh, yes, sure" I said, "That one!"

After the triumph of democracy in Spain, Federico came back to our country, first to the university of Barcelona, in 1978, where he inspired – as he did later in Madrid – the work of young mathematicians, just by talking of ideas and inviting people to express their own ideas. At that time, he was not formally advising any students, but if one traces the study of algebraic geometry in Barcelona and in Madrid, one will recognize the interests of Gaeta from Italy proliferating in South America, and later in the USA (the present volume is partially a proof of that assertion). Of course, in Barcelona he again got into trouble – his stay there has never been forgotten – especially when he was appointed Dean of the faculty, a position demanding much diplomacy, so unsuited to his direct character.

When he later came to Madrid, in 1982, I met him for the first time, and benefited much from his friendship and scientific motivation; but I always tried to avoid the kind of problems in which he continued to get involved ... till his retirement in 1988 (and even after!). But that was the way he was, and that was why we loved him.

With his passing we have lost a good friend. And mathematics, geometry in particular, has lost one of its last classical practitioners. And we Spaniards, with much less mathematical tradition than a number of other countries, have had the honor of his presence among us.

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Federico Gaeta and his wife, Teresa, in Rome

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Federico Gaeta and His Italian Heritage

Ciro Ciliberto

Abstract. This paper is devoted to give some information about the period prof. F. Gaeta spent in Italy as a research fellow of the Istituto Nazionale di Alta Matematica.

Mathematics Subject Classification (2000). 01A70; 14-03.

Keywords. Federico Gaeta, Istituto Nazionale di Alta Matematica, Italian Algebraic Goemeters, Spanish Algebraic Geometers.

1. Introduction

I have been asked by my friends and colleagues professors M. Alonso, E. Arrondo, R. Mallavibarrena and I. Sols of the Complutense University of Madrid to participate in the project of writing a book in honour of the late prof. Federico Gaeta. Given the respect and affection I have for Gaeta's memory, I accepted with enthusiasm. My task was to write a paper devoted to the years Gaeta spent in Italy as a student of F. Severi. I decided to reduce at a minimum the technical details, since other contributions to this volume will cover these aspects. I followed instead, without aiming at completeness, Gaeta's biography, centering on his Roman period. Since I was mostly interested in the first part of his scientific life, in which the connections with the Italian mathematical environment are more evident, this is the period on which I focused, limiting myself to shortly mentioning the rest. The reader may find additional information in the contributions by professor I. Sols to this volume. Unfortunately I have not been able to clarify all the biographical details I would have liked to reconstruct. In case of doubts, I will warn the reader. In particular, I tried to find, whenever possible, personal data for all the people mentioned in this article, but I did not succeed in all cases.

2. The beginnings

Federico Gaeta Maurelo was born in Madrid on March 3, 1923. He entered high school at the Institute "Cervantes" in Madrid and got his high school degree

This work was completed with the support of the G.N.S.A.G.A. of I.N.dA.M..

(Bachillerato) in 1940. In the same year he started the university courses at the Central University (Universidad Central) in Madrid, where he graduated, far as I could reconstruct, in 1943. The Central University of Madrid has been called Complutense University since 1970. R. Mallavibarrena says in [61], that Gaeta was then assistant (ayudante) at the same university in the years 1944–1950. In fact, it results from an official document sent to me by I. Sols, that Gaeta was assistant of "Análisis Matemático IV", but only in the years 1944–1946.

Francesco Severi

In a recent interview sent to me by I. Sols, Gaeta's widow, doña Teresa Aréjola (b. 1922), says that Gaeta went to Italy in 1943 with a fellowship, and he became there a student of F. Severi (1879–1961). The date 1943 is however doubtful and will be discussed below. How did Gaeta first develop an interest in algebraic geometry and how did he get in touch with Severi is difficult to say. However, algebraic geometry was highly reputed and sufficiently cultivated at that time in Spain. Moreover there were connections between Severi and the Spanish academy, probably motivated also by political reasons, namely the contiguity between Franco's dictatorship and the Fascist regime, of which Severi was a prominent representative. For example Severi was a member of the Spanish Real Mathematical Society (Real Sociedad Matemática Española) and he was invited more than once to Spain. In particular, he gave a series of talks in 1942 in Spain on the occasion of the third centennial of Galileo Galilei's (1564–1642) death. In the Revista Matematica Hispano-Americana [68], we read:

> *Las relaciones que ligan al sabio analista con el Instituto "Jorge Juan", alguno de cuyos Profesores trabajan bajo su dirección en el "R. Ist. Naz. di Alta Matematica", de Roma, se estrechan en estos momentos al figurar como huésped de honor de la Ciencia española.*
>
> [*The relationships between the famous analyst and the Institute "Jorge Juan", some of whose Professors work under his direction at the "R. Ist. Naz. di Alta Matematica", in Rome, become now stronger with his visit as a guest of honour of the Spanish science.*]

The "Jorge Juan" Mathematics Institute of Madrid, created by the Consejo Superior de Investigaciones Científicas (CSIC), was the most important mathematical research centre in Spain. It is possible that Gaeta attended Severi's talks there in 1942 and was introduced to him on that occasion.

Gaeta got his Ph.D. (Doctorado) at the Central University of Madrid around 1945. In [63], p. 351, one reads that T. Rodríguez Bachiller (1899–1980) was Gaeta's advisor and that the title of the thesis was "Análisis de las variaciones del grupo proyectivo sobre un cuerpo abstracto, respecto de las extensiones posibles de dicho cuerpo" (Analysis of the variations of the projective group over an abstract field, with respect to possible extensions of the field). It is probably the case that Gaeta's first paper [17] is related with his thesis. Bachiller, a very interesting and versatile character (see [43], p. 55), was not a full body algebraic geometer, though he visited Severi in Rome in 1941 and spent some time visiting O. Zariski (1899–1986) at the University of Illinois and S. Lefschetz (1884–1972), E. Artin (1898–1962) and C. Chevalley (1909–1984) at Princeton, where he also visited A. Einstein (1879–1955) in 1951 (see [43], l.c. and p. 329 of the 1946–47 annual report [63]).

Tomás Rodríguez Bachiller

Pedro Abellanas

It should be noted that the reports [63] witness an unsuspectable fervent mathematical activity, including algebraic geometry, in Spain in the years we are talking about. Besides the courses and seminars of Bachiller in Madrid, I like to mention the seminar in Zaragoza, where Gaeta will later become a full professor. They were held by P. Abellanas (1914–1999), a former student of Bachiller, and by others, and the subject were interesting classical and new topics in algebraic geometry (see, e.g., p. 353 of the 1945 CSIC annual report [63]).

3. The fellowship in Rome

I have not been able to find any official document that confirms doña Teresa Aréjola's memories about Gaeta coming to Italy in 1943. The first indication I found of Gaeta's presence in Italy is at p. 31 of the monograph [67] by G. Roghi on

the history of the Istituto Nazionale di Alta Matematica (INdAM) in Rome. There is mention there of the fact that Gaeta was nominated *research fellow* (discepolo ricercatore) at the Institute directed by F. Severi on January 4, 1946. This position, which did not imply any financial support from INdAM, was renovated for the years 1947–1950, and only in 1951 Gaeta was awarded with an INdAM fellowship, the "Lina Belluzzo" fellowship. This fellowship had been funded in 1940 by senator G. Belluzzo, with government security bonds worth 100,000 Italian lire. Because of the devaluation of the Italian currency due to the war, in 1951 the available fund was only of 39,316 lire. On p. 34 of [67], we read:

> *Il C.d.A. decide di arrotondare la cifra a* 50.000 *lire, conferendola come premio per attività scientifica svolta da un discepolo dell'Istituto, tenendo anche conto delle condizioni economiche del candidato; assegnatario di questa borsa è il dott. F. Gaeta, cittadino spagnolo.*
>
> [*The Administration Board decides to round up the sum to* 50,000 *lire, assigning it as a prize for the scientific activity of a fellow of the Institute, also taking into account the economical conditions of the candidate; the recipient of this fellowship is dr. F. Gaeta, a Spanish citizen.*]

The first time I found mention of a Spanish support to Gaeta for his stay in Italy, is in the 1946–47 CSIC annual report [63], on p. 329. It is written there that Gaeta was supported by a fellowship of the "Junta de Relaciones Culturales del Ministerio de Asuntos Exteriores" for the year 1947. The fellowship was extended to 1948 (see [63], p. 228). The extension was partially supported by the Italian Ministry of Foreign Affairs, due to the intervention of Severi who made a plea for it. I could not find any mention of other financial support to Gaeta for his stay in Rome from 1948 to 1951. This probably explains the not flourishing "economical conditions of the candidate" mentioned in the above INdAM report.

It is thus doubtful that Gaeta came to Italy in 1943 for a permanent stay. As I said, there seems to be no official mention of this. Moreover, how could have Gaeta reconciled a long stay abroad with his position as an assistant at the Central University in Madrid? Finally, as I shall later say, the years between 1943 and 1946 were extremely difficult for INdAM and for Severi himself, and it does not look likely he could have accepted in Roma the young Spanish student.

4. The theory of "liaison"

Once in Rome, Gaeta became a student of Severi. The research problem he got interested in arose, as usually at that time, from one of the courses of High Geometry (Alta Geometria) delivered by Severi at the Institute. As Gaeta says in the introduction of his second paper, during his course in the academic year 1946–47:

> *... il prof. Severi ha posto il concetto di residuale d'una curva algebrica sghemba irriducibile priva di punti multipli ... Il prof. Severi ha poi*

proposto di cercare quali sono le curve sghembe di residuale finito (*s'esse non son tutte*).

[*. . . prof. Severi defined the concept of residual of a skew, irreducible, algebraic curve with no multiple points. Then prof. Severi posed the question of determining all skew curves with a finite residual* (*if not all curves are like this*).]

According to Severi a smooth, irreducible curve $C \subset \mathbb{P}^3$ has *0 residual* if it is a complete intersection of two surfaces. The residual is then defined by induction: C has residual ρ if it is linked, in a complete intersection, to a curve with residual $\rho-1$. Therefore Severi's problem translates in the one of classifying all curves which can be linked, in a finite number of steps, to a complete intersection. The name of *liaison*, given by the French school, to the this concept, stands in fact for *linkage*.

Giuseppe Gherardelli

Oscar Zariski

Complete intersections are, in a sense, the easiest space curves one can consider, i.e., the ones which are defined by the *smallest number* of equations, namely two. A famous theorem, proved a few years before by G. Gherardelli (1894–1944) [42], an older student of Severi, provided a beautiful characterization of complete intersections. Curves with a finite residual appeared therefore the next easiest curves to be investigated, and this is probably the reason why Severi proposed the problem.

Gaeta was very fast in picking up this important question, and his great merit was to understand that both classical geometric and new algebraic techniques had to be used in order to solve it. In [18] he elegantly settled the classification problem for curves with residual $\rho = 1$. In [19, 20, 21, 22] he announced the complete solution of Severi's problem: curves with a finite residual coincide with the *arithmetically normal* ones. This is a concept introduced by O. Zariski in [71]: a projective variety is arithmetically normal if its *coordinate ring* is *integrally closed*. The geometric relevance of this concept, which linked it with very classical properties, had been pointed out by H.T. Muhly in [64]: a variety is arithmetically normal if and only if all linear series cut out on it by the hypersurfaces of the ambient projective space are complete. Related investigations were performed at the same time by the French school, first with M. Légaut (1900–1990) [57, 58, 59],

then P. Dubreil (1904–1994) with the concept of *varieties k-times of the first species* (variétés k-fois de première espèce) in [11, 12, 13], and by R. Apéry (1916–1994) in [5, 6]. A famous cohomological characterization of arithmetical normality and of varieties k-times of the first species, which are essentially, in modern terms, the so-called *arithmetically Cohen-Macaulay varieties*, will be later given by J.P. Serre (b. 1926) in his epochal paper [69], which opens up a new era in algebraic geometry.

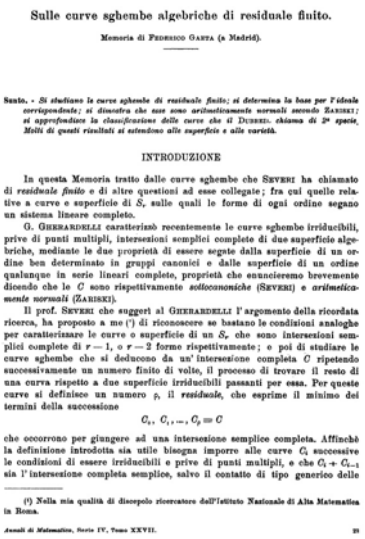
Arithmetical normality entered also in the aforementioned theorem of Gherardelli, which says that: a curve in $\mathbb{P}^3$ is a complete intersection if and only if it is arithmetically normal and *subcanonical*.

Paul Dubreil in 1954

Wolfgang Gröbner

The first complete exposition of Gaeta's theory, including a quick proof of Gherardelli's theorem and applications to surfaces in higher dimensional projective spaces, appears in the memoir [23]. This is followed by the short note [24], in which Gaeta announces a new series of results, which are the core of his contribution to the subject, namely the description of the ideal generated by of all homogeneous polynomials vanishing on an arithmetically normal curve in $\mathbb{P}^3$. This theory is fully developed in the subsequent memoir [25], followed by a series of short notes [27, 30, 31, 32, 33, 34, 35] in which refinements and complements can be found.

In [25] Gaeta uses the classical theory of *syzigies* by D. Hilbert (1862–1943) and F.S. Macaulay (1862–1937) (see [48, 60]), revived by Dubreil in [14] and W. Gröbner (1899–1980) [44, 45], to show that the ideals in question are *determinantal ideals*, precisely they are generated by the minors of maximal order of suitable matrices of type $(\rho + 1) \times (\rho + 2)$ of homogeneous polynomials, where ρ is the residual of the curve. From the degrees of the entries of the matrix, several algebraic and geometric properties of the curves can be read off. Gaeta also shows that similar results hold for finite sets of points in the plane. As a consequence, he proves a long-standing conjecture, which went back to the very beginning of the theory of space curves, with G. Halphen (1844–1889) and M. Noether (1844–1921) (see [47, 66]), asserting that the number of invariants which are needed to classify all families of space curves is an increasing function of the degree. In modern terms this means that the number of components of the *Hilbert scheme* of curves in $\mathbb{P}^3$ grows with high rate with respect to the degree (for a more recent version of this result, in the spirit of Gaeta's approach, see [15]).



Sulle curve sghembe algebriche di residuale finito.

Memoria di FEDERICO GAETA (a Madrid).

Sunto. - *Si studiano le curve sghembe di residuale finito; si determina la base per l'ideale corrispondente; si dimostra che esse sono aritmeticamente normali secondo* ZARISKI*; si approfondisce la classificazione delle curve che il* DUBREIL *chiama di 2ª specie. Molti di questi risultati si estendono alle superficie e alle varietà.*

INTRODUZIONE

In questa Memoria tratto dalle curve sghembe che SEVERI ha chiamato di *residuale finito* e di altre questioni ad esse collegate; fra cui quelle relative a curve e superficie di S_r sulle quali le forme di ogni ordine segano un sistema lineare completo.

G. GHERARDELLI caratterizzò recentemente le curve sghembe irriducibili, prive di punti multipli, intersezioni semplici complete di due superficie algebriche, mediante le due proprietà di essere segate dalla superficie di un ordine ben determinato in gruppi canonici e dalle superficie di un ordine qualunque in serie lineari complete, proprietà che enunceremo brevemente dicendo che le C sono rispettivamente *sottocanoniche* (SEVERI) e *aritmeticamente normali* (ZARISKI).

Il prof. SEVERI che suggerì al GHERARDELLI l'argomento della ricordata ricerca, ha proposto a me (1) di riconoscere se bastano le condizioni analoghe per caratterizzare le curve o superficie di un S_r che sono intersezioni semplici complete di $r-1$, o $r-2$ forme rispettivamente; e poi di studiare le curve sghembe che si deducono da un' intersezione completa C ripetendo successivamente un numero finito di volte, il processo di trovare il resto di una curva rispetto a due superficie irriducibili passanti per essa. Per queste curve si definisce un numero ρ, il *residuale*, che esprime il minimo dei termini della successione

$$C_i,\ C_{i+1},\ C_\rho \equiv C$$

che occorrono per giungere ad una intersezione semplice completa. Affinchè la definizione introdotta sia utile bisogna imporre alle curve C_i successive le condizioni di essere irriducibili e prive di punti multipli, e che $C_i + C_{i-1}$ sia l' intersezione completa semplice, salvo il contatto di tipo generico delle

(1) Nella mia qualità di discepolo ricercatore dell'Istituto Nazionale di Alta Matematica in Roma.

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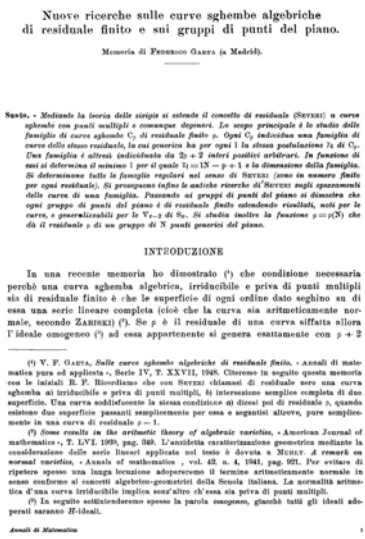


Nuove ricerche sulle curve sghembe algebriche di residuale finito e sui gruppi di punti del piano.

Memoria di FEDERICO GAETA (a Madrid).

Sunto. - *Mediante la teoria delle serie speciali si estende il concetto di residuale* (SEVERI) *a curve sghembe con punti multipli e comunque degeneri. Lo scopo principale è lo studio delle famiglie di curve sghembe C_ρ di residuale finito ρ. Ogni C_ρ individua una famiglia di curve dello stesso residuale, la cui generica ha per ogni l la stessa postulazione γ_l di C_ρ. Una famiglia è altresì individuata da $2\rho+2$ interi positivi arbitrari. In funzione di essi si determina il minimo l per il quale $\gamma_l = lN - p + 1$ e la dimensione della famiglia. Si determinano tutte le famiglie regolari nel senso di* SEVERI *(sono in numero finito per ogni residuale). Si proseguono infine le antiche ricerche di* SEVERI *sugli spezzamenti delle curve di una famiglia. Passando ai gruppi di punti del piano si dimostra che ogni gruppo di punti del piano è di residuale finito adeguatamente risultati, noti per le curve, e generalizzabili per le V_{r-2} di S_r. Si studia inoltre la funzione $\rho = \rho(N)$ che dà il residuale ρ di un gruppo di N punti generici del piano.*

INTRODUZIONE

In una recente memoria ho dimostrato (1) che condizione necessaria perchè una curva sghemba algebrica, irriducibile e priva di punti multipli sia di residuale finito è che le superficie di ogni ordine dato seghino su di essa una serie lineare completa (cioè che la curva sia aritmeticamente normale, secondo ZARISKI) (2). Se ρ è il residuale di una curva siffatta allora l'ideale omogeneo (3) ad essa appartenente si genera esattamente con $\rho+2$

(1) V. F. GAETA, *Sulle curve sghembe algebriche di residuale finito*, « Annali di matematica pura ed applicata », Serie IV, T. XXVII, 1948. Citeremo in seguito questa memoria con le iniziali R. F. Ricordiamo che con SEVERI chiamasi di residuale zero una curva sghemba ai irriducibile e priva di punti multipli, b) intersezione semplice completa di due superficie. Una curva soddisfacente la stessa condizione a) dicesi poi di residuale ρ, quando esistono due superficie passanti semplicemente per essa e segantisi altrove, pure semplicemente in una curva di residuale $\rho-1$.

(2) *Some results in the arithmetic theory of algebraic varieties*, « American Journal of mathematics », T. LVI, 1939, pag. 340. L'accennata caratterizzazione geometrica mediante la considerazione delle serie lineari applicata nel testo è dovuta a MUMFORD. *A remark on normal varieties*, « Annals of mathematics », vol. 42, n. 4, 1941, pag. 521. Per evitare di ripetere spesso una lunga locuzione adopereremo il termine aritmeticamente normale in senso conforme ai concetti algebrico-geometrici della Scuola Italiana. La normalità aritmetica d'una curva irriducibile implica senz'altro ch'essa sia priva di punti multipli.

(3) In seguito sottintenderemo spesso la parola *omogeneo*, giacchè tutti gli ideali adoperati saranno H-ideali.

Annali di Matematica 1

First page of Gaeta's paper [23]

First page of Gaeta's paper [25]

As mentioned in the 1948 CSIC annual report [63], p. 228, the memoir [25] was awarded a prize from the "Sección de Ciencias Exactas, Física y Naturales" of the CSIC.

I will not dwell more on Gaeta's results on "liaison" here, since they are the object of a more thorough analysis in the contribution of prof. R. Miró-Roig to the present volume. I should stress however the originality and modernity of Gaeta's approach. Though he certainly owed a lot to Severi, who proposed him the question and taught him the *Italian way* of looking at the matter, it has to be acknowledged to Gaeta the essential idea, uncommon inside the Italian school, of using homological and commutative algebra in purely geometric investigations. The rich interplay between algebra and geometry which can be found in Gaeta's papers and their clear and transparent writing style still make them a very stimulating reading for young persons, as they have been for people of my generation.

Max Noether

Georges Halphen

5. Elliptic surfaces

Gaeta's research in his Italian period is not exhausted by the results on "liaison". There are in fact three interesting papers on elliptic surfaces [26, 28, 29] to be mentioned. The first one is a preliminary note, in which all results of the remaining two are quickly exposed.

The theory of surfaces was considered, within the Italian school, to be one of the main achievements of the Italian science. An algebraic geometer was hardly considered to be of first class, unless he had been able to give some serious contribution to this theory. It was therefore natural that Gaeta had to prove his skills also in this field in order to reach an excellence level among the algebraic geometers of his generation.

Guido Castelnuovo

Federigo Enriques

As is well known, F. Enriques (1871–1946) and G. Castelnuovo (1865–1952) set up, in the last decade of XIX century, the foundations of the theory, proved some of the main results and outlined the classification using the values of the *plurigenera*. Severi entered the scene only a bit later, since he graduated in Torino with C. Segre (1863–1924) in 1900. However, his contributions to the subject have been relevant, first in collaboration with Enriques, then alone, after the harsh contrasts which arose between them in the 1920s (see [9]). Taking advantage from the prominent position he reached within the Fascist regime, Severi aimed at attributing to himself most of the merit of the great construction of the theory. In the period we are talking about, Enriques having left the scene in 1946 and Castelnuovo being old and retired, Severi was the unopposed leader of the school. One of his objectives was to pursue a revision of the theory, especially in the parts in which Enriques and his school left an undisputed trace. One of these was the classification of elliptic surfaces, which, together with brilliant ideas, presented gaps and imperfections. This, I think, is the origin of the papers [26, 28, 29], whose subject was very likely suggested by Severi. In them one finds a nice and more precise presentation of Enriques theory, especially for the classification of regular elliptic surfaces. The theory of elliptic surfaces, among other things, has been completely re-established in the 1960's by K. Kodaira (1915–1997) in his epochal papers [50, 51, 52, 53, 54, 55, 56].

6. The Istituto Nazionale di Alta Matematica

I think it is now the right moment to make an excursus on the life of the Istituto Nazionale di Alta Matematica in the years Gaeta visited it. This will give me the opportunity of shedding some light on the atmosphere that Gaeta found during his stay in Rome from 1946 to 1952.

The Mathematics Department "G. Castelnuovo" of the University of Roma "La Sapienza", where INdAM is located

The Istituto Nazionale di Alta Matematica was founded on July 13, 1939, and F. Severi was soon nominated its President. We are talking about one of the most obscure periods of the recent Italian history. I do not think I can depict the situation better and more briefly than G. Roghi does on p. 11 of [67]:

> *Occorre non dimenticare l'anno, il* 1939*, cui si riferiscono questi episodi; le leggi razziali sono dell'anno precedente; nel mese di settembre è iniziata la seconda guerra mondiale con l'Italia che assume una ambigua posizione di "non belligeranza" essendo alleata alla Germania dal cosiddetto Patto d'Acciaio.*
>
> *Questa atmosfera non poteva non avvertirsi nella vita degli atenei e dell'Indam. È esemplare che in vari verbali del C.S. i professori Bompiani e Picone ... vengano indicati come i "camerati Bompiani e Picone", che fra i documenti richiesti dal C.d. A. per partecipare ai concorsi per assistente, per borsista, ecc., figuri il "certificato di iscrizione al P.N.F." (Partito Nazionale Fascista), nonché una dichiarazione di "appartenenza alla razza ariana".*
>
> [*One should not forget the year,* 1939*, in which this happened: the racial laws had been promulgated the year before; in September the second world war started with Italy assuming an ambiguous position of "non belligerence", though being an ally of Germany with the so called Iron Treaty.*

This atmosphere was perceived also in the daily life of the universities and at Indam. It is remarkable that, in various reports of the C.S. [*Scientific Committee*]*, professors Bompiani and Picone ... are indicated as "comrades Bompiani and Picone", that, among the documents required by the C.d.A.* [*Administration Board*] *in order to apply for positions of assistants, for fellowships, etc., appears the "enrollment certificate to the P.N.F."* (*National Fascist Party*)*, as well as a declaration of "belonging to the Aryan race".*]

At this time, within the Roman mathematical ambient, Severi was an undisputed leader: Enriques, the old adversary, had been defeated and then expelled from the university with the racial laws, Castelnuovo retired in 1936, thus avoiding the shame of the expulsion, G. Scorza (1876–1939), another prominent algebraist and algebraic geometer, certainly not unpopular to the regime, he was in fact a Senator, died in the same year in which INdAM was born.

To describe how tough the situation was, I will recall an episode reported to me by prof. A. Franchetta (b. 1916). When the racial laws where promulgated, Enriques was even forbidden to go to the University, to visit the library and to borrow books. Was then Scorza who, outraged by this shameful decision, promised Enriques to borrow from the library the books he would have needed to consult and to take them to his place.

The Istituto di Alta Matematica was, in practice, a creation of Severi and a reward given to him by the Fascist regime for his loyalty. This was stressed by the presence of Benito Mussolini himself at the inauguration and at the celebration of the first year of activity of the Institute. The institute however started its activity in a very difficult moment, not only from the general political viewpoint. At that time the situation of science, in particular of mathematics, in Italy was not at all flourishing. This has been well explained by various authors (see, e.g., [8, 49, 65]) and therefore I will not enter into details here. Sticking to mathematics, it will be sufficient to recall that the racial laws literally devastated the Italian academy, expelling scientists of the calibre of F. Enriques, G. Fano (1871–1952), B. Levi (1875–1961), T. Levi-Civita (1873–1941), B. Segre (1903–1977), A. Terracini (1889–1968), to mention only a few.

Roma, April 15, 1940: B. Mussolini at INdAM listens, with G. Bottai, Severi's speech for the celebration of the first year of activity of the Institute

Severi however, was not the kind of person who was satisfied only with his personal power and prestige. He was a top scientist, he cared about the institution and therefore he was willing to do his best for promoting research and creating an internationally highly reputed centre. So, despite the scarcely reassuring setting in which INdAM was created, the Institute was able to have positive effects on the Roman mathematical ambient and to help young people to come across with excellent mathematicians, with good research projects, in a collaborative and helpful atmosphere, from which Gaeta had the chance of taking advantage (see [65, 70]).

The activity at the Institute consisted in:

- the courses given by F. Severi, professor of High Geometry (Alta Geometria), L. Fantappiè (1901–1956), professor of High Analysis (Alta Analisi) and G. Krall (1901–1971), professor of Applications of High Mathematics (Applicazioni di Alta Matematica);
- courses occasionally given by other professors of the University of Rome;
- lectures and seminars given by visiting professors coming from other Italian or foreign universities.

Good young mathematicians had the honorary position of *research fellow*, which gave them the right of freely participating in the activities of the Institute and using its facilities. In practice this was a position given to well reputed young assistants at the University of Rome or to people, like Gaeta, whose stay in Rome was supported by fellowships given by different institutions. Fellowships of INdAM were given to particularly promising young students that Severi wanted to attract to Rome.

On the whole, in the first years of its life, INdAM was the centre of a reasonable amount of scientific activity, including an International Congress which took place in November 1942, just in the middle of the second world war. This congress had the rather bold pretence of being a substitute of the International Congress of Mathematicians, which should have taken place in 1940, and did not just because of the outburst of the war. Of course only mathematicians from allied countries of Italy were invited, and moreover, as one could imagine, as a consequence of the aberrant racial laws, no Jewish mathematician, like Castelnuovo and Enriques, could participate (for more details and comments, see [67], p. 20–21).

The Institute, as well as the whole country, undergoes a deep crisis in 1943, after the fascist regime collapsed (July 25, 1943), and the civil war divided Italy in two, the south and the centre occupied by the Anglo-Americans, the north under the puppet regime of the so called Salò Republic controlled by the Germans. Severi was suspended by the Ministry of Public Education (Ministro della Pubblica Istruzione) from his position of President of INdAM on August 16, 1944, a position which was formally restituted to him only about four years later, on July 20, 1948. He was however able to informally run the Institute since 1946 (see [67]). Besides the suspension, Severi was also subject to a purge in the years 1944–1945, because of his involvement with the fascist regime. He was also suspended from teaching in that period. The "purge" was then commuted into an official "reproach" in 1945, but Severi could not go back to teaching till 1946.

Basically the life of the Institute was totally stopped for about two years, from 1943 to 1945, and very little activity was present in 1945. This is the reason why it is very unlikely that Gaeta came to Rome in a stable way before January 1946, the first moment in which there is mention of him, in the documents at INdAM, as a research fellow. Starting from 1946, the scientific life of the Institute gradually restarted. Just sticking to the activity close to geometry in the years from 1946 to 1952, which most likely Gaeta attended, we have:

Beniamino Segre in 1960

Lucio Lombardo Radice

- Severi's course of High Geometry, in the academic years 1945–46, 1946–47, 1947–48 (devoted to "Continuous systems of curves on a surface and related integrals"), 1948–49, 1949–50 (devoted to "Theory of abelian integrals on varieties"), 1950–51, 1951–52 (devoted to "Simple and multiple algebraic integrals");
- short courses given by L. Lombardo Radice (1916–1982) and F. Conforto (1909–1954) in the academic year 1946–47; by G. Vaccaro (1917–2004) on topics in Topology in 1949 and on the "Theory of homotopic groups" in 1950; by B. Segre (on "Harmonic integrals") and by F. Conforto (on "Abelian functions") in 1951; by B. Segre (on "Differential forms and their integrals") and by F. Conforto (on "Modular abelian functions") in 1952;
- talks given by various mathematicians like B. Segre in May 1947; R. Nevanlinna (1895–1980), B. Segre in April 1948; W. Blaschke (1885–1962), L. Godeaux (1887–1975), W. Gröbner, G. Polya (1887–1985), L. Roth (1904–1968), B. Segre in 1949; W. Blaschke, L. Brower (1881–1966), P. Dubreil, L. Godeaux, W. Gröbner, H. Hasse (1898–1979), J. Leray (1906–1998), B. Segre in 1950; B. Finzi (1899–1974), W. Blaschke, L. Roth, G. Sansone (1888–1979), B. Segre, G. Vaccaro, in 1951; R. Caccioppoli (1904–1959), W. Blaschke, G. Julia (1893–1978), E. Kähler (1906–2000) in 1952.
- an International Conference on Geometry, in honour of Severi on the occasion of the 50th anniversary of his university degree, in the period April 26–28, 1950, where the following mathematicians were invited: A.S. Besicowitch (1891–1970), R. Behnke (1898–1979), W. Blaschke, L.E.J. Brower, E. Čech (1893–1960), M. Deuring (1907–1984), P. Dubreil, A. Duschek (1895–1957),

L. Godeaux, R. Garnier (1887–1984)), W. Gröbner, H. Hasse, H. Hopf (1894–1971), K. Kuratowski (1896–1980), L.J. Mordell (1888–1972), T.K. Pöschl (1882–1955), G. de Rahm (1903–1990), H.S. Ruse (1905–1974), L. Roth, J.A. Schouten (1883–1971) J.G. Semple (1904–1985), W. Sierpinski (1882–1969), J.L. Synge (1897–1995), B.L. van der Waerden (1903–1996).

Bert van der Waerden

Leonard Roth

Jack Semple

This long list shows how increasingly lively, much more than today, the activity of the Institute became in the years in which Gaeta visited it. Certainly Gaeta took profit of it, and this had an enormous influence on his mathematical education.

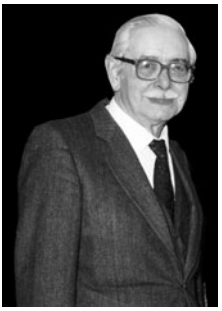
Notice P. Dubreil, W. Gröbner and B.L. van der Waerden among the visitors of INdAM in that period. It is very likely that Gaeta had the chance of discussing with them about his problem, getting suggestions about the use of algebraic and homological methods in order to solve it. Actually, soon after Rome, in 1952–53, Gaeta visited Dubreil with a fellowship in Paris.

It is interesting to put down also a list of young Italian algebraic geometers or foreign researchers who have been fellows of INdAM at the same time Gaeta was there (the full list can be found in [67]): A. Andreotti (1924–1980), I. Barsotti (1921–1987), M. Benedicty, W. Burau (1906–1944), J. Casulleras (1920–1996), V. Dalla Volta (1918–1982), G. Dantoni (1909–2005), A. Franchetta, D. Gallarati (b. 1923), G. Gherardelli (1925–2008), H. Haefli, W. Haefli, C. Longo (1912–1971), J. Molina, J. Sebastiao e Silva (1914–1972), L. Lombardo Radice, E. Martinelli (1911–1998), L. Nollet, G. Pompilj (1913–1968), J. Ribeiro de Albuquerque (1910–1991), M. Rosati (b. 1928), T. Ruiz de Pablo, G. Saban (b. 1926), F. Succi (b. 1924), J. Teixidor (1920–1989), G. Vaccaro, E. Vesentini (b. 1928), G. Zappa (b. 1915).

With some of them, like Andreotti and Franchetta, Gaeta had very good relationships, both from the personal and scientific viewpoint.

The scientific interaction was particularly solid with Andreotti, the youngest, very brilliant student of Severi, probably due to the fact of being both Severi's favorites at that time. For example Gaeta mentions a private communication by Andreotti in [28], whereas Andreotti often quotes Gaeta in his famous papers on the classification of irregular surfaces: in [1] he refers to Gaeta's memoir [25],

Aldo Andreotti: the INdAM period



Alfredo Franchetta

in [2] he quotes [28], in [3] again [25] is quoted, finally in [4] the paper [26] is mentioned. The citation in [3] is particularly important, since Andreotti proposes there a stimulating extension of Gaeta's techniques to the classification of irregular surfaces in abelian varieties defined by determinantal equations in theta functions: a promising viewpoint which has never been more deeply investigated afterwards.

7. The chair in Zaragoza

On November 27, 1950, F Gaeta won a competition (oposición) for a chair of Projective and Descriptive Geometry at the University of Zaragoza, with Abellanas, who in 1940 moved to Madrid, in the committee. Doña Teresa Aréjola remembers that Severi wrote a recommendation letter for Gaeta. She also points out that Gaeta was not hired in Zaragoza till 1953. The cause of the delay is not clear. Doña Teresa suggests that personal and temperamental reasons entered in this matter. Certainly Gaeta was not an easy character, and his free and independent attitude very badly complied with the rigid and conformist schemes of Franco's regime. Let me mention a famous story, first reported to me by prof. A. Franchetta, which I also read in §4 of [41], partly cited below, dedicated by the physicist A. Galindo Tixaire (b. 1934), President of the Spanish Real Academy of Sciences (Real Academia Nacional de Ciencias), to his former professor F. Gaeta:

> *... es verídico que como burla ante los méritos extra-científicos que algunos opositores presentaban en aquellos tiempos de imperante nacional-catolicismo él lo hiciera mostrando al tribunal una "benedicción papal" de esas que se podían comprar por un puñado de liras en cualquier estanco de Roma.*
>
> [*... it is true that, as a joke, as extra-scientific merits that some of the candidates presented in those times of generalized national-catholicism, he submitted to the committee a "papal blessing", one that could be bought for a few liras in some shop in Rome.*]

It is likely that Gaeta was disappointed by the delay in coming into possession of his chair. However he did not waste time meanwhile. As we saw, he spent

one more academic year 1951–52 in Rome and after that, one in Paris, visiting P. Dubreil.

Finally he got his position in Zaragoza in 1953. His first period there was very active. In the 1952–53 CSIC annual report [63], p. 896, we read a summary of his activity in 1953: besides the publication of various papers mentioned above, he gives courses on recent developments of algebraic geometry. However, shortly, his relationships with his mathematicians colleagues in Zaragoza became very tense. Prof. Franchetta tells me that once Gaeta reported to him about harsh contrasts, for political reasons, with the faculty. Once more I can refer to [41], who, commenting in a footnote upon the heavy atmosphere which at that time reigned at the faculty of Sciences of Zaragoza, says:

> *Hasta en la orla de fin de carrera de mi promoción se aprecia el desastre. Figuran en la ella los Prof. García Atance, Arajuo, Iñiguez, Rodrígues-Salinas, Cabrera, Servera, Liso, Rodrígues Vidal, Burbano y Gaeta, pero este último nos pidió estar lejos de sus colegas, entre los alumnos ...*
>
> [*Even from the photograph taken at the end of the courses of my class one can appreciate the disaster. Appear there Prof. García Atance, Arajuo, Iñiguez, Rodrígues-Salinas, Cabrera, Servera, Liso, Rodrígues Vidal, Burbano and Gaeta, but the latter asked us to stay far from his colleagues, among the students ...*]

Another striking testimony of this tension, not only at a local, but even at a national and international level, is given by what happened at the first Congress of the "Groupement de Mathématiciens d'Expression Latine", which took place in 1957 in Nice. I want to report about this because it also shows the close connections that still linked Gaeta with the Italian ambient at that time. Connections also witnessed by various visits to Italy and talks given at INdAM and in Bari (see [67, 36]). On p. 203–204 of [62] we read:

> *Com hem dit, la trobada de Niça consistia en nou conferències plenàries. Cada conferència anava seguida de discussió, oberta pels comentaris de dos* animateurs. *El pes de la presència espanyola, mesurada en termes de conferenciants, "animadors" i presidents de les sessions, era discret, i possiblement reflectia bé el pes de les matemàtiques espanyoles relativament als del altres països. Tres espanyols: San Juan, Abellanas i Ríos havien estat invitats a fer "d'animadors", però cap no havia estat honrat amb la presidència d'una sessió. Gaeta, claramanet un* outsider *respecte a la Comunitat Matemàtica Espanyola, havia estat invitat a donar una confèrencia plenària. Entre els matemàtics espanyols, algú va proposar aixecar-se ostensiblement i abandonar la sala de conferències quan Gaeta comencés la seva exposició. Sunyer va negar-s'hi, trencant el consens que semblava imposar-se. Finalment ningú non va a abandonar la sala per la confèrencia de Gaeta.*

Ferran Sunyer i Balaguer

Enrique Vidal Abascal was mathematician and painter: selfportrait

[*As we said, the congress in Nice consisted of nine plenary talks. Each talk was followed by a discussion, which was open by the comments of the* animateurs. *The weight of the Spanish presence, measured in terms of speakers, "animators" and chairmen was fairly good, and possibly well reflected the value of the Spanish mathematics with respect to the ones of the other countries. Three Spanish mathematicians: San Juan, Abellanas and Ríos had been invited as "animators", but no one had the honour of being a chairman. Gaeta, clearly an* outsider *with respect to the Spanish Mathematical Community, had been invited to deliver a plenary talk. Among the Spanish mathematicians, somebody proposed to visibly rise up and leave the congress room when Gaeta was to start his exposition. Sunyer disagreed, thus breaking the consent that seemed to stand out. At the end nobody left the room for Gaeta's talk.*]

The "animator" of Gaeta's talk was Andreotti.

The Spanish delegation at the congress, besides Gaeta, was formed by P. Abellanas (Central University of Madrid), F. Sunyer i Balaguer (1912–1967, University of Barcelona), G. Ancochea (1908–1981, Central University of Madrid), S. Ríos (1913–2008, Central University of Madrid), J. Teixidor (University of Barcelona), E. Vidal Abascal (1908–1994, University of Santiago de Compostela). R. San Juan (1908–1969, Central University of Madrid) had been invited but could not participate. Also the Catalan L. Santaló (1911–2001, Buenos Aires University) was invited as a special guest from the organizing committee.

The other algebraic geometers, besides Gaeta, who had the honour of delivering a plenary talk, were B. Segre and Severi, who gave the closing lecture. The importance of these speakers shows how well reputed was Gaeta inside the international scientific community. It also shows how much Severi was still influential at that time.

Going back to the unpleasant atmosphere Gaeta had to face in Zaragoza, it was clear that this situation could not last too long. As Galindo remembers in [41], despite a petition to the Rector of the University of Zaragoza signed by Gaeta's

affectionate students, in 1956 Gaeta was moved by authority to the University of Santiago de Compostela. As prof. J. Vaquer i Timoner (b. 1928) points out in the recent interview mentioned at the end of this paper, at that time the only Spanish universities with a curriculum in mathematics, were Barcelona, Madrid and Zaragoza. In other universities there were courses in mathematics, but only as a subsidiary discipline for other degrees, like chemistry. In particular in Santiago there was no chair in Projective Geometry and Vaquer says that some professors in Santiago got outraged by the decision of the central authorities of sending Gaeta there, arguing that their university was not to be considered as penal colony. In any event, Gaeta never took possession of his new, undesired position. Indeed, the rector of the university allowed him to live in Zaragoza, since he had no teaching duties in Santiago. In practice, Gaeta was excluded from teaching, almost the worst that could have happened. Worse than that, excluding physical attacks or intimidations, only destitution could have been contemplated. This, however, was not out of discussion, since, just a few years later, in 1965, eminent professors, like J.L. Aranguren (1909–1996), A.G. Calvo (b. 1926), and E.T. Galván (1918–1986), of the Complutense University, were in fact expelled by the regime and A. Navarro (1916–1992) and S. Montero Díaz (1911–1985), of the same university, were suspended for two years. The reason was the fact that they took the leadership of a great student manifestation against the regime which took place on the campus of the Central University. In solidarity with his colleagues, the poet J.M. Valverde (1926–1996) also resigned from his chair of Aesthetics at the University of Barcelona, declaring that "Nulla aesthetica sine ethica", i.e., "There is no aesthetics without ethics".

I think it was useful to dwell on these events to reconstruct, at least partially, the atmosphere of those years in Spain, and to support the idea that Gaeta has to be considered as a precursor of the uneasiness that part of the Spanish academy felt against the regime. In any event, unlike the cases mentioned above, it is difficult to understand from written testimonies like [41], the gravity of the events that led to the disciplinary measure against Gaeta. This is a piece of history which has still to be written and certainly it goes well beyond the scope of the present article. I will limit myself to register here that certainly this was probably the main reasons of Gaeta's departure from Spain in 1957.

Prof. Vaquer remembers that Gaeta sent from abroad his resignation letter to a friend who never forwarded it. By contrast C. París, a philosopher who had been a colleague of Gaeta in Zaragoza, writes (see [16]):

> *Federico Gaeta, aunque no se encontraba en el ejercicio de su cátedra, sino como profesor en los Estados Unidos, renunció también, desde allí, a su condición de catedrático y me hizo llegar copia del divertido telegrama que había enviado a Lora Tamayo, a la sazón Ministro de Educación y Ciencia, en que, tras comunicar su renuncia, terminaba con una rotunda frase: "Aprovecho la ocasión para expresarle mi mís absoluto desprecio".*

[*Federico Gaeta, though no longer in possession of his chair, but as professor in the United States, resigned from his chair and let me receive a copy of the amusing telegram that he sent to Lora Tamayo, who was at the time Minister of the Education and Science, in which, after having communicated his resignation, he finished with the sharp sentence: "I take the chance of expressing to you my absolute contempt".*]

From which we may understand part of his frank and outspoken character.

8. Exile and return

Having decided to leave Spain, it would have been more than natural for Gaeta to look for a position in Italy. May be he even did, but I have no information about that. However at the time the Italian system was rigid enough to make this quite difficult. In fact, one could be hired as a full professor in an Italian University only if he won a national competition (concorso). There were rather few of these competitions, and the Italian academy was provincial enough to look at foreign candidates with suspect, if not with hostility. In any event, Gaeta first got positions for three years in Argentina (Buenos Aires, La Plata, Bariloche), then for three more years in Brasil (San Paulo), for ten months in Venezuela (Caracas), finally in the United States at the New York State University at Buffalo, where he remained for fifteen years, till 1977. During his stay in Buffalo he hosted for several months in his house A. Grothendieck (b. 1928) and wrote the notes of a course [46] that Grothendieck gave at the University of Buffalo.

Alexandre Grothendieck

According to doña Teresa Aréjola, Gaeta, in the years he spent abroad, used to say he would have gone back to Spain after Franco's death. This in fact happened. Franco died in 1975, and Gaeta got a chair at the University of Barcelona in 1977. In the 1980 yearbook of the University of Barcelona he is mentioned as the Dean of the Faculty of Mathematics. Prof. Vaquer remembers that he was a good dean. At that time, members of the faculty started speaking Catalan during

the meetings (Juntes de Facultat). Though this was not yet allowed by the law, which was imposing the use of Castilian in public meetings, Gaeta never objected. The only hesitation he had with Catalan was that he did not like to examine a Catalan speaking student alone: not that he was unable to understand him, but he did not want that the student could argue that he was unable to understand.

In 1982, he finally moved back to his origins, namely to the Complutense University in Madrid, where he retired in 1992, becoming then emeritus in 1993. In Madrid he died on April 4, 2007.

During the years abroad Gaeta's contacts with Italy became less intense, but they saw a revival from the second half of the 1970's on. In 1974 he published again in an Italian journal, fifteen years after [36], the paper [37], soon followed by [38, 39, 40]. The former notes were presented at the Accademia dei Lincei by B. Segre. The latter is Gaeta's first note in Italian after seventeen years and it is the text of a talk he gave in Milan. From the late 1970's Gaeta started again visiting Italy, something he loved to do. He especially visited more often and for longer periods Rome and Milan. For Rome of course he had a special affection. In Milan he had a good friend, prof. E. Marchionna (1921–1993), whom he knew since his first period in Italy: note that both, Gaeta and Marchionna were in Taormina in 1951, for the IV Congress of the Italian Mathematical Union, where they gave a communication (see [35]).

Ermanno Marchionna

Gaeta's scientific horizon was now, after so many years from his first stay in Italy, different from the past, however it was still quite classical and very close to the Italian tradition: he was now mainly interested in Riemann surfaces and their jacobians and to geometric applications of invariant theory. His contributions to these topics are not comparable with the breakthrough in the classification of space curves carried out by him in the 1950's. Still his papers are interesting and, above all, well written and nice to read. Again, for people of my generation, they have been a source of information about the classical state of the art, of problems and of inspiration.

Certainly I am now quite far from the limit I gave myself, namely the illustration of Gaeta's Italian period. Therefore it is time for me to stop. Though I think the few information about Gaeta after he left Italy in 1952 can be interesting to better understand his character and how strong was the footprint of the Italian scientific education for him.

9. A few personal considerations and memories

I want to finish with a few personal comments and memories about F. Gaeta.

The six years in which Gaeta visited Rome were certainly not an easy time. Italy was beginning to emerge from the darkness of twenty years of dictatorship and of a devastating war. Somebody makes jokes about the fact that Italians have been able to lose and win that war at the same time. Italy however paid a very high price for this, including the civil war and seeing its territory occupied and devastated by opposite foreign troops. The year in which Gaeta arrived in Rome, 1946, was the one in which the country started recovering from the wounds of a hatred past. From a political viewpoint 1946 was marked by lively public and private political debates and by the first free elections after the fascist dictatorship which decreed the election of the Constitutional Assembly (Assemblea Costituente) and the end of the monarchy. In 1946, though poverty and discomfort were still largely spread out among the population, the reconstruction started. The atmosphere was therefore a positive one and strong was the hope in a better future. Certainly Gaeta had the chance of breathing this new air, so different from the one he could have experienced in Franco's Spain. And he could see, in the six years he spent in Italy, the good fruits of the new situation. It is likely this indelibly influenced his way of thinking, in mathematics and in other matters.

I have been a student of prof. A. Franchetta, who has been in turn, together with G. Pompilj, the last student of F. Enriques. When Enriques suddenly died in 1946, despite the antagonism between Enriques and Severi, the latter took Franchetta, a young and brilliant algebraic geometer, under his protection. Thus Gaeta and Franchetta, though the latter was a few years older, have been fellow pupils of Severi. The brilliant character of the young, lively and outspoken Spanish came often to the stage during the descriptions Franchetta made to me, more than once, about his young days. Remembrances that I had the chance to revive with him just a few days ago when I told him that I was about to write this article. So, in a sense, through Franchetta's descriptions, I met Gaeta before I had the pleasure of knowing him personally. Moreover, Gaeta's main papers [23, 25] of that period had been mentioned to me by Franchetta and they were in his files in the office we shared at the University of Naples. So I had the chance of studying them just at the beginning of my admittance into the algebro-geometric world. I remember it was a pleasure to read so clearly written papers, in which the power of the geometric intuition was enhanced with algebraic techniques. A rather infrequent case among the classical algebraic geometers of the Italian school,

and this made them particularly original. It was the reading of these papers that convinced me commutative and homological algebra are basic tools which any algebraic geometers should always have in his tool kit.

I met personally Gaeta in 1978, at the International Congress of Mathematicians in Helsinki. I well remember the occasion: we were both sailing on a boat for the usual congress trip and have been introduced to each other by common colleagues. This was the time in which he was going back to Spain after the long period abroad, and he was restarting his visits to Italy. Since then we met several times both in Italy and in Spain. He often visited Rome and I had the chance of discussing with him about mathematics and many other topics. He was a lively and curious person, no way one could get bored in his company. In 1990 he participated at the "Combinatorics" conference in the town of Gaeta, on the coast, half way between Naples and Rome. I met him in Rome on his way to the conference, and he was very amused by the idea of visiting the place from which his family name originated. I was invited to Madrid to give a talk a few years later, and Gaeta and his wife took me by car for a very nice day-long trip to Toledo. Gaeta's driving was not exactly impeccable, but all the rest was fine. At the end of the day he made a present to me: a booklet with the history of Toledo that I still have as a very nice souvenir in my personal library. In 1998 I had the pleasure of being invited to Madrid as a speaker to a conference in his honour.

Federico Gaeta, as well as other mathematicians of his generations mentioned here, build for us, in not easy conditions, important pathways connecting classic with modern algebraic geometry. For this reason, and also for their own results, some of them fundamental, we owe them a deep gratitude. Gaeta's death deprived us of one of the last representative of this generation.

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Part II

Linkage Theory

This part has been coordinated by Enrique Arrondo with the advice of Rosa María Miró-Roig. It is devoted to linkage theory and contains, besides a survey, five research articles.

Federico Gaeta (on the right) in his Italian period

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Gaeta's Work on Liaison Theory: An Appreciation

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Abstract. This short note is a survey of Gaeta's results on Liaison theory.

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Keywords. Liaison, curve, Cohen-Macaulay.

Federico Gaeta Maurelo, Professor of Geometry at the Universidad Complutense de Madrid, Spain, died on April 7, 2007, at the age of 84. Born on March 3, 1923, in Madrid, his talent for Mathematics and interest in Geometry showed at an early age. Don Federico, as he was known, was not only a fine mathematician, but also a man with a deep interest in culture.

Don Federico was passionate about mathematics and had an endless curiosity. He published more than 45 scientific works, most of them in the area of Algebraic Geometry. In his series of papers, he addressed problems of Projective Geometry, Invariant Geometry, Liaison Theory, Commutative and Computational Algebra (Gröbner basis), Elimination Theory and Schottky Problem, among others. I will focus my attention on his contributions to Liaison Theory.

In 1946, F. Gaeta was enrolled in the "Istituto Nazionale di Alta Matematica" in Roma where he worked under the direction of Professor F. Severi and where he wrote his substantial contributions to Liaison Theory. As F. Gaeta says in his work [12], Professor F. Severi suggested him to characterize space curves with finite residual (i.e., to characterize licci curves in $\mathbb{P}^3$). F. Gaeta solved this problem in a masterly and elegant way; this was a seminal work in Liaison Theory and his most important contribution to this active area of research. Historically Liaison began in the nineteenth century as a tool to study and classify curves in projective spaces and it goes back at least to work of M. Noether, F. Severi and F.S. Macaulay. The initial idea was to start with a curve in $\mathbb{P}^3$ and to study its residual in a complete intersection. It turns out that a lot of information can be carried over from a

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curve to its residual and vice versa. It was hoped that using this process of linking curves one could always pass to a "simpler" curve, namely a complete intersection (in some sense the simplest curve), and so it would be possible to get information about the original curve from this complete intersection.

In the first half of the twentieth century, P. Dubreil [5], R. Apéry [1]–[3] and F. Gaeta [7]–[15], among others, contributed to the initial development of Liaison Theory. To begin with, let us recall the definition of curve with finite residual as it was introduced by F. Severi (see also [12]; Chap. III, §16).

Definition 1. An irreducible non-singular curve $C \subset \mathbb{P}^r$ is said to be a *curve of residual* 0 if it constitutes the complete simple intersection of $r-1$ hypersurfaces, and is said to be a *curve of residual* ρ, $\rho > 0$, if there exist $r-1$ hypersurfaces whose complete intersection consists of C and an irreducible nonsingular curve C_1 of residual $\rho - 1$, each component being simple.

Nowadays, we say:

Definition 2. Two subschemes V_1 and V_2 of $\mathbb{P}^n$ are said to be (*algebraically*) *directly CI-linked* by a complete intersection subscheme $X \subset \mathbb{P}^n$ if $I(X) \subset I(V_1) \cap I(V_2)$ and we have $I(X) : I(V_1) = I(V_2)$ and $I(X) : I(V_2) = I(V_1)$.

If V_1 and V_2 do not share any common component the above definition has a clear geometric meaning. In fact, it is equivalent to $X = V_1 \cup V_2$ as schemes and in this case we say that V_1 and V_2 are directly *geometrically CI-linked*.

Let V_1 and $V_2 \subset \mathbb{P}^n$ be two equidimensional schemes without embedded components. We say that V_1 and V_2 are in the same *CI-liaison class* if there exists a sequence of schemes $Y_1, \ldots, Y_r$ such that Y_i is *directly CI-linked* to Y_{i+1} and such that $Y_1 = V_1$ and $Y_r = V_2$. In other words *CI-Liaison* is the equivalence relation generated by directly CI-linkage and roughly speaking Liaison theory is the study of these equivalence relations and the corresponding equivalence classes.

Remark 3. A curve $C \subset \mathbb{P}^r$ has finite residual if it can be linked in a finite number of steps to a complete intersection (shortly, if it is *licci)*.

Example. A simple example of curve of residual 1 is the following one: let C_1 be the twisted cubic in $\mathbb{P}^3$ with homogeneous ideal

$$I(C_1) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subset K[x_0, x_1, x_2, x_3],$$

and let C_2 be the secant line to C_1 defined by $I(C_2) = (x_1, x_2)$. The union of C_1 and C_2 is the degree 4 curve X with ideal

$$I(X) = (x_1^2 - x_2x_0 + x_2^2 - x_1x_3, x_1^2 - x_0x_2 + 2x_2^2 - 2x_1x_3).$$

Therefore, X is a complete simple intersection of two quadrics. Since C_2 is a line, it has residual 0 and we conclude that C_1 has residual 1 or, equivalently, C_1 is directly CI-linked to a complete intersection. In particular, C_1 is licci.

It was hoped that all curves had finite residual and the first example of space curve which had not finite residual was given by R.A. Apéry in [1]. Classical Liaison Theory had as a main goal to characterize curves with finite residual. In [12], F. Gaeta solved this problem in a brilliant and elegant way; he characterized the curves $C \subset \mathbb{P}^3$ with finite residual and he proved that they are precisely the arithmetically normal curves. As a first step towards the characterization of space curves with finite residual, he proved that an irreducible non-singular curve $C \subset \mathbb{P}^3$ has residual 1 if the ideal of homogeneous polynomials vanishing simply on C has a base consisting of exactly three forms (see [7] and [12]; Chap. III, §22 for a classification of space curves of residual 1). Later, he obtained a full characterization of space curves with finite residual after a deep study of arithmetically normal curves. Irreducible, arithmetically normal varieties $V \subset \mathbb{P}^r$ were first studied by O. Zariski and by H.T. Muhly.

In [27], O. Zariski introduced the following definition:

Definition 4. An irreducible algebraic variety $V \subset \mathbb{P}^r$ is said to be *normal* (in the arithmetic sense) if the ring of the homogeneous coordinates of its general point is integrally closed in its quotient field.

Remark 5. O. Zariski proved that if $V \subset \mathbb{P}^n$ is a normal (in the arithmetic sense) variety, then the linear system of hyperplane sections is complete. In other words, a normal (in the arithmetic sense) variety is also normal in the geometric sense, as it was used by the Italian geometers.

From now on, I will call *arithmetically normal variety* to any variety which is normal in the arithmetic sense. In 1941, H.T. Muhly gave the following nice characterization of arithmetically normal algebraic varieties.

Theorem 6. *Let $V_r \subset \mathbb{P}^n$ be an r-dimensional algebraic variety. A necessary and sufficient condition for V_r to be arithmetically normal is that for every integer m the linear system cut out on V_r by the hypersurfaces of degree m in $\mathbb{P}^n$ is complete.*

Proof. See [23]; p. 291. □

Using in a clever way Muhly's characterization of arithmetically normal varieties, F. Gaeta proved:

Proposition 7. *Let $C, C' \subset \mathbb{P}^r$ be two irreducible non-singular curves. Assume that $C \cup C'$ is the complete simple intersection of $r-1$ hypersurfaces. Then, C is arithmetically normal if and only if C' is arithmetically normal.*

Nowadays, we have:

Proposition 8. *Let $X, Y \subset \mathbb{P}^n$ be two equidimensional locally Cohen-Macaulay subschemes of the same dimension $d \geq 1$. Assume that X and Y are directly CI-linked by Z. Then X is arithmetically Cohen-Macaulay if and only if Y is arithmetically Cohen-Macaulay.*

Proof. See, for instance, [22]; Corollary 2.1.4. □

Proposition 7 was a first step towards the characterization of curves with finite residual. In [9], a paper full of beautiful and deep geometrical results, F. Gaeta posed the following question:

Question 9. *Let $C \subset \mathbb{P}^n$ be an arithmetically normal curve. Is C a curve of finite residual?*

For $n = 3$, F. Gaeta answered positively to the above question and he got his main contribution to Liaison Theory, namely:

Theorem 10. *Let $C \subset \mathbb{P}^3$ be a curve. C has finite residual if and only if C is arithmetically normal. In addition, the ideal of polynomials vanishing on a curve C of residual ρ consists of exactly $\rho + 2$ surfaces.*

Proof. See [9] and [12]; Chap. III, §25 and §26. For a modern proof of this result the reader can also look at [25]. □

F. Gaeta generalized his result and showed: Let $X, Y \subset \mathbb{P}^r$, $r \geq 4$, be two smooth irreducible surfaces. Assume that $X \cup Y$ is the complete simple intersection of $r - 2$ hypersurfaces. Then, X is arithmetically normal if and only if Y is arithmetically normal. To prove it he used the following lemma:

Lemma 11. *Let $X \subset \mathbb{P}^r$ be a smooth irreducible surface. A necessary and sufficient condition to X to be arithmetically normal is that its generic hyperplane section $X \cap H$ is arithmetically normal.*

Proof. See [12]; Chap. II; §10. □

As a corollary, he got:

Corollary 12. *Let $S \subset \mathbb{P}^r$, $r \geq 4$, be a surface of finite residual. Then, S is arithmetically normal.*

Proof. See [9] and [12]; Chap. II; §9. □

As an application of Proposition 7 and Corollary 12, F. Gaeta proved the existence of arithmetically normal subcanonical curves $C \subset \mathbb{P}^r$, $r \geq 4$, (resp. surfaces $S \subset \mathbb{P}^r$, $r \geq 5$) which are not complete simple intersection of $r - 1$ (resp. $r - 2$) hypersurfaces (see [12]). Nowadays, this result comes from the fact that in codimension greater or equal to 3 it is not true that arithmetically Gorenstein schemes and complete intersection schemes coincide.

Theorem 10 was immediately extended by F. Gaeta in [13] and R. Apéry in [4] to arithmetically normal varieties V_{n-2} of codimension 2 in $\mathbb{P}^n$, $n \geq 3$. Indeed, they defined:

Definition 13. A *matrix variety* V_{n-2} in $\mathbb{P}^n$, $n \geq 3$, is a variety of dimension $n - 2$ defined by the vanishing of all the h-rowed determinants of a matrix (A_{ij}), $i = 1, \ldots, h$; $j = 1, \ldots, h + 1$, the A_{ij} being homogeneous polynomials of suitably restricted degrees in the coordinates $x_0, \ldots, x_n$.

F. Gaeta in [13] and R. Apéry in [4] proved:

Theorem 14. *Every variety $V_{n-2} \subset \mathbb{P}^n$ of dimension $n-2$ which can be defined by a chain of complete simple intersections, starting from a complete intersection of primals, is a matrix variety V_{n-2}; and vice versa any matrix variety $V_{n-2} \subset \mathbb{P}^n$ of dimension $n-2$ is arithmetically normal and it can be defined by a chain of intersections, starting from a complete simple intersection of $n-2$ forms.*

Proof. See [13]. □

Remark 15. (a) More generally, a *matrix variety* V_{n-c} in $\mathbb{P}^n$ is a variety of dimension $n-c$ defined by the vanishing of all the h-rowed determinants of a matrix (A_{ij}), $i = 1, \ldots, h$; $j = 1, \ldots, h+c-1$, the A_{ij} being homogeneous polynomials of suitably restricted degrees in the coordinates $x_0, \ldots, x_n$. Therefore, a matrix variety is nothing but a *standard determinantal* variety (i.e., codimension c varieties defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix).

(b) Thanks to Hilbert-Burch Theorem we know that matrix varieties (i.e., standard determinantal varieties) $V_{n-2} \subset \mathbb{P}^n$ and codimension 2 arithmetically normal subvarieties of $\mathbb{P}^n$ coincide. In codimension $c \geq 3$, it is no longer true that arithmetically normal varieties and matrix varieties (i.e., standard determinantal varieties) of dimension $n-c$ coincide. For instance, let $X = \{p_1, \ldots, p_{10}\} \subset \mathbb{P}^3$ be a set of 10 general points. X is an arithmetically normal variety but X is not standard determinantal. Indeed, the only way to define a 0-dimensional variety $X \subset \mathbb{P}^3$ of length 10 as the maximal minors of a $t \times (t+2)$ homogeneous matrix is by means of a 3×5 matrix with linear entries. Therefore, the dimension of the family of 0-dimensional schemes $X \subset \mathbb{P}^3$ of length 10 defined by the maximal minors of a 3×5 matrix with linear entries is 27 while the dimension of the family of sets of 10 general points in $\mathbb{P}^3$ is 30.

Nowadays, Theorem 14 can be stated as follows;

Theorem 16. *Let $V \subset \mathbb{P}^n$ be a codimension 2 subscheme. V is a standard determinantal scheme (i.e., it is defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix) if and only if it is CI-linked in a finite number of steps to a complete intersection (i.e., V is licci).*

We are led to pose the following question:

Question 17. *Let $V_{n-c} \subset \mathbb{P}^n$ be a matrix variety of codimension c. Can V_{n-c} be obtained by a chain of complete simple intersections, starting from a complete simple intersection of $n-c$ forms?*

The answer to the above question as well as to Question 9 is no, in general. In other words, in the classical context of Liaison Theory, Gaeta's Theorem does not generalize to varieties of higher codimension. See [18] or [19]–[22] for more information.

Going on with the problem of classifying space curves, in [13] and [14], F. Gaeta made important advances in the problem of classifying space curves with

finite residual or, equivalently, in the problem of classifying arithmetically normal curves in $\mathbb{P}^3$. Using the Theory of syzygies developed by D. Hilbert and V. Gröbner, he established the following theorems (see [13]; §2):

(a) An arithmetically normal curve $C \subset \mathbb{P}^3$ determines an integer $\rho \geq 0$ such that the ideal of polynomials vanishing on C is generated by the $(\rho+1)$ maximal minors of a homogeneous matrix $\mathcal{A}$ of $\rho+1$ rows and $\rho+2$ columns, the general element c_{ij} of which is a homogeneous polynomial of order $\mu_{ij} = \mu_{i1} + \mu_{ij} - \mu_{11}$, which may be taken in a normal form in which $\mu_{11} \leq \mu_{12} \leq \cdots \leq \mu_{1,\rho+2}$, $\mu_{11} \geq \mu_{21} \geq \cdots \geq \mu_{\rho+1,1}$.

(b) If f_i is the minor obtained by suppressing the ith column of $\mathcal{A}$, then the surfaces $f_1 = 0$, $f_2 = 0$ meet, residually to C, in an arithmetically normal curve whose corresponding matrix is obtained by deleting the first two columns of $\mathcal{A}$ and transposing.

Whence,

(c) The curve C is of finite residual ρ.

Remark 18. It follows from the previous results that the ideal of polynomials vanishing on a curve $C \subset \mathbb{P}^3$ of residual ρ has a base consisting of $(\rho+2)$ forms (see [12]; §24). The converse is true for $\rho = 0$ or 1 and it turns out to be false for $\rho \geq 2$. In fact, in [12], F. Gaeta pointed out the existence of space curves C which have not residual 2 and such that the ideal of polynomials vanishing on C consists of exactly 4 forms. For example: the rational quartic C in $\mathbb{P}^3$.

Many of these last results extend to V_{n-2} in $\mathbb{P}^n$ and to sets of points in a plane. For instance, the 0-dimensional version of Theorem 14 was treated by F. Gaeta in [16]. He proved that the homogeneous polynomial ideal defined by N generic points in $\mathbb{P}^2$ is generated by the determinants of order $\rho+1$ $(\rho > 0)$ of a $(\rho+1)\times(\rho+2)$ homogeneous matrix $\mathcal{A}$. This matrix is supposed normalised so that the degrees of the polynomials decrease down the first column and increase along the first row. Considerations of homogeneity determine the degree of an element of the matrix in terms of elements in the first row and first column. F. Gaeta proved that if the N points in $\mathbb{P}^2$ are generic then all the entries of the matrix $\mathcal{A}$ are homogeneous polynomials of degree one or two and, in addition, he described a simple algorithm by which, N being given, ρ and the degrees of the polynomials may be determined. More precisely, we have:

Theorem 19. *Set $R = k[x,y,z]$ and $\mathbb{P}^2 = Proj(R)$. For any integer $s \geq 2$, there exists a non-empty subset $U_s \subset (\mathbb{P}^2)^s$ such that for any $X = \{p_1, \ldots, p_s\} \in U_s$, the ideal $I(X)$ of X has a locally free resolution of the following type:*

$$0 \longrightarrow R(-d-2)^a \oplus R(-d-1)^b \longrightarrow R(-d-1)^c \oplus R(-d)^d \longrightarrow I(X) \longrightarrow 0$$

where $\binom{d+1}{2} \leq s < \binom{d+2}{2}$ and $bc = 0$.

Proof. See [16]. For a modern proof of this result the reader can also look at [17]; Theorem 2.6. □

In 1974, C. Peskine and L. Szpiro gave a rigorous proof of Gaeta's Theorem: every arithmetically Cohen-Macaulay codimension 2 subscheme X of $\mathbb{P}^n$ can be CI-linked in a finite number of steps to a complete intersection subscheme; i.e., X is licci (see [25]). They also established Liaison Theory as a modern discipline. Roughly speaking, Liaison is an equivalence relation among subschemes of a given dimension d in a projective space $\mathbb{P}^n$ (or ideals in $K[x_0, x_1, \ldots, x_n]$) and it involves the study of the properties that are shared by two schemes whose union is well understood. For recent results on Liaison Theory the reader can look at [18], [19], [20], [21], [22] and the references quoted there.

As we have seen, Liaison theory has its roots dating more than 60 years ago and, although the greatest activity has been in the last 30 years, F. Gaeta brought substantial contributions to Liaison Theory which will long last for their importance and geometrical beauty.

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Symmetric Ladders and G-biliaison

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Abstract. We study the family of ideals generated by minors of mixed size contained in a ladder of a symmetric matrix from the point of view of liaison theory. We prove that they can be obtained from ideals of linear forms by ascending G-biliaison. In particular, they are glicci.

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Introduction

Ideals generated by minors have been studied extensively. They are a central topic in commutative algebra, where they have been investigated mainly using Gröbner bases and combinatorial techniques (see among others [10], [18], [1], [2], [21], [17]). They are also relevant in algebraic geometry, since many classical varieties such as the Veronese and the Segre variety are cut out by minors. Degeneracy loci of morphisms between direct sums of line bundles over projective space have a determinantal description, as do the Schubert varieties.

In this paper, we study ideals of minors in a symmetric matrix from the point of view of liaison theory. In particular, we consider ideals generated by minors of mixed size which are contained in a symmetric ladder. Cogenerated ideals in a ladder of a symmetric matrix belong to the family that we study. The family of cogenerated ideals is a natural one to study from the combinatorial point of view (see [7] or [8]). However, from the point of view of liaison theory it is more natural to study a larger class of ideals, as they naturally arise during the linkage process. We call them symmetric mixed ladder determinantal ideals.

In Section 1 we set the notation and define symmetric mixed ladder determinantal ideals (Definition 1.3). In Example 1.5 (3) we discuss why cogenerated ladder determinantal ideals of a symmetric matrix are a special case of symmetric mixed ladder determinantal ideals. In Proposition 1.7 we show that symmetric

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mixed ladder determinantal ideals are prime and Cohen-Macaulay. In Proposition 1.8 we express their height as the cardinality of a suitable subladder.

In Section 2 we review the notion of G-biliaison, stating the definition and main result in the algebraic language (see Definition 2.2 and Theorem 2.3). In Theorem 2.4 we prove that symmetrix mixed ladder determinantal ideals can be obtained from ideals of linear forms by ascending G-biliaison. In particular, they are glicci (Corollary 2.5).

1. Ideals of minors of a symmetric matrix

Let K be an algebraically closed field. Let $X = (x_{ij})$ be an $n \times n$ symmetric matrix of indeterminates. In other words, the entries x_{ij} with $i \leq j$ are distinct indeterminates, and $x_{ij} = x_{ji}$ for $i > j$. Let $K[X] = K[x_{ij} \mid 1 \leq i \leq j \leq n]$ be the polynomial ring associated to the matrix X. In this paper, we study ideals generated by the minors contained in a ladder of a generic symmetric matrix from the point of view of liaison theory. Throughout the paper, we only consider symmetric ladders. This can be done without loss of generality, since the ideal generated by the minors in a ladder of a symmetric matrix coincides with the ideal generated by the minors in the smallest symmetric ladder containing it.

Definition 1.1. A **ladder** $\mathcal{L}$ of X is a subset of the set $\mathcal{X} = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i,j \leq n\}$ with the following properties :

1. if $(i,j) \in \mathcal{L}$ then $(j,i) \in \mathcal{L}$ (i.e., $\mathcal{L}$ is symmetric), and
2. if $i < h, j > k$ and $(i,j),(h,k) \in \mathcal{L}$, then $(i,k),(i,h),(h,j),(j,k) \in \mathcal{L}$.

We do not make any connectedness assumption on the ladder $\mathcal{L}$. For ease of notation, we also do not assume that X is the smallest symmetric matrix containing $\mathcal{L}$. Let

$$\mathcal{X}^+ = \{(i,j) \in \mathcal{X} \mid 1 \leq i \leq j \leq n\} \quad \text{and} \quad \mathcal{L}^+ = \mathcal{L} \cap \mathcal{X}^+.$$

Since $\mathcal{L}$ is symmetric, $\mathcal{L}^+$ determines $\mathcal{L}$ and viceversa. We will abuse terminology and call $\mathcal{L}^+$ a ladder. Observe that $\mathcal{L}^+$ can be written as

$$\mathcal{L}^+ = \{(i,j) \in \mathcal{X}^+ \mid i \leq c_l \text{ or } j \leq d_l \text{ for } l = 1,\dots,r \text{ and}$$
$$i \geq a_l \text{ or } j \geq b_l \text{ for } l = 1,\dots,u\}$$

for some integers $1 \leq a_1 < \cdots < a_u \leq n$, $n \geq b_1 > \cdots > b_u \geq 1$, $1 \leq c_1 < \cdots < c_r \leq n$, and $n \geq d_1 > \cdots > d_r \geq 1$, with $a_l \leq b_l$ for $l = 1,\dots,u$ and $c_l \leq d_l$ for $l = 1,\dots,r$.

The points $(a_1,b_2),\dots,(a_{u-1},b_u)$ are the **lower outside corners** of the ladder, $(a_1,b_1),\dots,(a_u,b_u)$ are the **lower inside corners**, $(c_2,d_1),\dots,(c_r,d_{r-1})$ the **upper outside corners**, and $(c_1,d_1),\dots,(c_r,d_r)$ the **upper inside corners**. If $a_u \neq b_u$, then (a_u,a_u) is a lower outside corner and we set $b_{u+1} = a_u$. Similarly, if $c_r \neq d_r$ then (d_r,d_r) is an upper outside corner, and we set $c_{r+1} = d_r$. See also Figure 1. A ladder has at least one upper and one lower outside corner. Moreover, $(a_1,b_1) = (c_1,d_1)$ is both an upper and a lower inside corner.

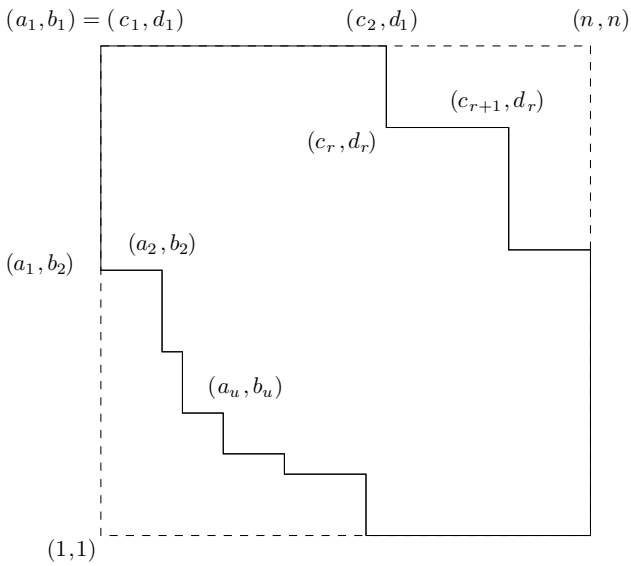

FIGURE 1. An example of ladder with tagged lower and upper corners.

The **upper border** of $\mathcal{L}^+$ consists of the elements (c, d) of $\mathcal{L}^+$ such that either $c_l \leq c \leq c_{l+1}$ and $d = d_l$, or $c = c_l$ and $d_l \leq d \leq d_{l-1}$ for some l. See Figure 2.

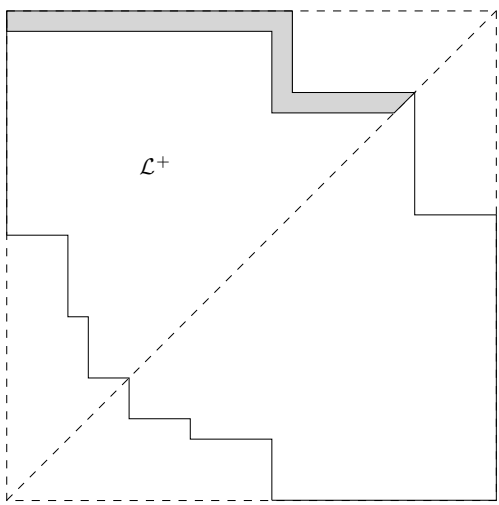

FIGURE 2. The upper border of the same ladder.

All the corners belong to $\mathcal{L}^+$. In fact, the ladder $\mathcal{L}^+$ corresponds to its set of lower and upper outside (or equivalently lower and upper inside) corners. The upper corners of a ladder belong to its upper border.

Given a ladder $\mathcal{L}$ we set $L = \{x_{ij} \in X \mid (i,j) \in \mathcal{L}^+\}$, and denote by $K[L]$ the polynomial ring $K[x_{ij} \mid x_{ij} \in L]$. For t a positive integer, and $1 \leq \alpha_1 \leq \cdots \leq \alpha_t \leq n$, $1 \leq \beta_1 \leq \cdots \leq \beta_t \leq n$ integers, we denote by $[\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_t]$ the t-minor $\det(x_{\alpha_i,\beta_j})$. We let $I_t(L)$ denote the ideal generated by the set of the t-minors of X which involve only indeterminates of L. In particular $I_t(X)$ is the ideal of $K[X]$ generated by the minors of X of size $t \times t$.

In this article, we study the G-biliaison class of a large family of ideals generated by minors in a ladder of a symmetric matrix.

Notation 1.2. Let $\mathcal{L}^+$ be a ladder. For $(c,d) \in \mathcal{L}^+$ let

$$\mathcal{L}^+_{(c,d)} = \{(i,j) \in \mathcal{L}^+ \mid i \leq c,\ j \leq d\}, \qquad L_{(c,d)} = \{x_{ij} \in X \mid (i,j) \in \mathcal{L}^+_{(c,d)}\}.$$

See also Figure 3. Notice that $\mathcal{L}^+_{(c,d)}$ is a ladder according to Definition 1.1 and

$$\mathcal{L}^+ = \bigcup_{(c,d)\in\mathcal{U}} \mathcal{L}^+_{(c,d)}$$

where $\mathcal{U}$ denotes the set of upper outside corners of $\mathcal{L}^+$.

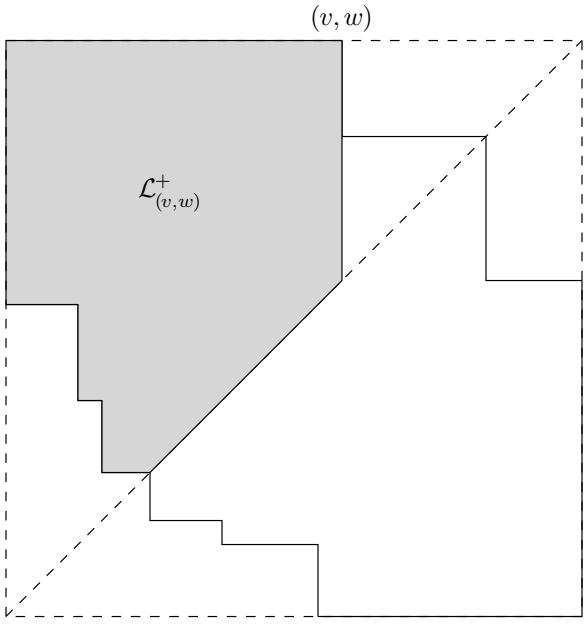

FIGURE 3. The ladder $\mathcal{L}^+$ with a shaded subladder $\mathcal{L}^+_{(v,w)}$.

Definition 1.3. Let $\{(v_1, w_1), \ldots, (v_s, w_s)\}$ be a subset of the upper border of $\mathcal{L}^+$ which contains all the upper outside corners. We order them so that $1 \leq v_1 \leq \cdots \leq v_s \leq n$ and $n \geq w_1 \geq \cdots \geq w_s \geq 1$. Let $t = (t_1, \ldots, t_s)$ be a vector of positive integers. Denote $L_{(v_k, w_k)}$ by L_k. The ideal

$$I_t(L) = I_{t_1}(L_1) + \cdots + I_{t_s}(L_s)$$

is a **symmetric mixed ladder determinantal ideal**. Denote $I_{(t,\ldots,t)}(L)$ by $I_t(L)$. We call $(v_1, w_1), \ldots, (v_s, w_s)$ **distinguished points** of $\mathcal{L}^+$.

Remarks 1.4.

1. Let $\mathcal{M} \supseteq \mathcal{L}$ be two ladders of $\mathcal{X}$, and let M, L be the corresponding sets of indeterminates. We have isomorphisms of graded K-algebras
$$\begin{aligned} K[L]/I_t(L) &\cong K[M]/I_t(L) + (x_{ij} \mid x_{ij} \in M \setminus L) \\ &\cong K[X]/I_{2t}(L) + (x_{ij} \mid x_{ij} \in X \setminus L). \end{aligned}$$
Here $I_t(L)$ is regarded as an ideal in $K[L], K[M]$, and $K[X]$ respectively. Then the height of the ideal $I_t(L)$ and the property of being prime, Cohen-Macaulay, Gorenstein, Gorenstein in codimension $\leq c$ (see Definition 2.1) do not depend on whether we regard it as an ideal of $K[L], K[M]$, or $K[X]$.
2. We can assume without loss of generality that for each $l = 1, \ldots, s$ there exists a $k \in \{1, \ldots, u-1\}$ such that
$$t_l \leq \min\{v_l - a_k + 1, w_l - b_{k+1} + 1\}$$
In fact, if $t_l > \min\{v_l - a_k + 1, w_l - b_{k+1} + 1\}$ for all k, then $I_{t_l}(L_l) = 0$. If that is the case, replace L by $M := \cup_{i \neq l} L_i$, eliminate (v_l, w_l) from the distinguished points and remove the lth entry of t to get a new vector m. Then we obtain a new ladder for which the assumption is satisfied and such that $I_m(M) = I_t(L)$.
3. We can assume that
$$w_k - w_{k-1} < t_k - t_{k-1} < v_k - v_{k-1}, \quad \text{for } k = 2, \ldots, s.$$
In fact, if $v_k - v_{k-1} \leq t_k - t_{k-1}$, by successively developing a t_k-minor of L_k with respect to the first $v_k - v_{k-1}$ rows we obtain an expression of the minor as a combination of minors of size $t_k - (v_k - v_{k-1}) \geq t_{k-1}$ that involve only indeterminates from L_{k-1}. Therefore $I_{t_k}(L_k) \supseteq I_{t_{k-1}}(L_{k-1})$. Similarly, if $w_k - w_{k-1} \geq t_k - t_{k-1}$, by developing a t_{k-1}-minor of L_{k-1} with respect to the last $w_{k-1} - w_k$ columns we obtain an expression of the minor as a combination of minors of size $t_{k-1} - (w_{k-1} - w_k) \geq t_k$ that involve only indeterminates from L_k. Therefore $I_{t_{k-1}}(L_{k-1}) \subseteq I_{t_k}(L_k)$. In either case, we can remove a part of the ladder and reduce to the study of a proper subladder that corresponds to the same symmetric ladder determinantal ideal.
4. We can always find $k \in \{1, \ldots, s\}$ such that $v_k > v_{k-1}$ and $w_k > w_{k+1}$. In fact, the two inequalities are satisfied if and only if (v_k, w_k) is an upper outside corner. Notice that if we have distinguished points (v_k, w_k) and (v_{k+1}, w_{k+1}) on the same row or column, then one of the following holds:

- either $v_k = v_{k+1}$ and $t_k > t_{k+1}$,
- or $w_k = w_{k+1}$ and $t_{k+1} > t_k$.

In particular, we can find $k \in \{1, \dots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$, unless $t_k = 1$ for all k.

The following are examples of determinantal ideals of a symmetric matrix which belong to the class of ideals that we study.

Examples 1.5.

1. If $t = (t, \dots, t)$ then $I_t(L)$ is the ideal generated by the t-minors of X that involve only indeterminates from L. These ideals have been studied in [4], [5], and [6].
2. If $\mathcal{L} = \mathcal{X}$, then according to Remarks 1.4 we can assume that $w_l = n$ for all $l = 1, \dots, s$ and $v_s = n$. From Remark 1.4 (3), we have $t_l > t_{l-1}$ and $v_l > v_{l-1}$ for all l. Then $I_t(L)$ is generated by the t_1-minors of the first v_1 rows, the t_2-minors of the first v_2 rows, ..., the t_s-minors of the whole matrix. This is a simple example of a cogenerated ideal.
3. The family of symmetric mixed ladder determinantal ideals contains the family of **cogenerated ideals** in a ladder of a symmetric matrix, as defined in [4]. We follow the notation of [4], and assume for ease of notation that $(1, n)$ is an inside corner of $\mathcal{L}$ (i.e., that X is the smallest matrix containing L). If $\alpha = \{\alpha_1, \dots, \alpha_t\}$, then $I_\alpha(L) = I_\tau(L)$ where $\{(v_1, w_1), \dots, (v_s, w_s)\}$ consists of the upper outside corners of $\mathcal{L}$, together with the points of the upper border of $\mathcal{L}$ which belongs to row $\alpha_l - 1$, for all l for which such an intersection point is unique (if for some l the intersection of the row $\alpha_l - 1$ with the upper border of $\mathcal{L}$ consists of more than one point, then $\mathcal{L}$ has an upper outside corner on the row $\alpha_l - 1$ and we do not add any extra point to the set). For each $k = 1, \dots, s$, we let $\tau_k = \min\{l \mid \alpha_l > v_k\}$.
4. Let X be a matrix of size $m \times n$, $m \leq n$, whose entries are indeterminates. Assume that X contains a square symmetric submatrix of indeterminates, and that all the other entries of X are distinct indeterminates. In block notation
$$X = \begin{pmatrix} M & N \\ S & P \end{pmatrix}$$
where S is a symmetric matrix of indeterminates and M, N, P are generic matrices of indeterminates. Let $t \in \mathbb{Z}_+$. Then $I_t(X)$ is a symmetric ladder determinantal ideal generated by the minors of size $t \times t$ contained in a symmetric ladder of
$$\begin{pmatrix} Y & M & N \\ M^t & S & P \\ N^t & P^t & Z \end{pmatrix}$$
where Y, Z are symmetric matrices of indeterminates, and M^t denotes the transpose of M. This was observed by Conca in [4].

In this section we establish some properties of symmetric mixed ladder determinantal ideals. It is known ([4]) that cogenerated ideals are prime and Cohen-Macaulay. In the sequel we show that the result of Conca easily extends to symmetric mixed ladder determinantal ideals. We exploit a well-known localization technique (see [3], Lemma 7.3.3). The same argument was used to prove Lemma 1.19 in [12]. For completeness, we state it for the case of a ladder of a symmetric matrix and we outline the proof. We use the notation of Definitions 1.1 and 1.3. From Remark 1.4 (4) we know that we can always find $k \in \{1, \dots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$, unless $t = (1, \dots, 1)$.

Lemma 1.6. *Let $\mathcal{L}$ be a ladder of a symmetric matrix X of indeterminates. $\mathcal{L}$ has a set of distinguished points $\{(v_1, w_1), \dots, (v_s, w_s)\} \in \mathcal{L}^+$ and $t = (t_1, \dots, t_s) \in \mathbb{Z}_+^s$. Let $I_t(L)$ be the corresponding symmetric mixed ladder determinantal ideal. Let $k \in \{1, \dots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$.*

Let $t' = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s)$ and let $\mathcal{L}'$ be the ladder obtained from $\mathcal{L}$ by removing the entries $(v_{k-1} + 1, w_k), \dots, (v_k - 1, w_k), (v_k, w_k), (v_k, w_k - 1) \dots, (v_k, w_{k+1} + 1)$ and the symmetric ones. Let

$$(v_1, w_1), \dots, (v_{k-1}, v_{k-1}), (v_k - 1, w_k - 1), (v_{k+1}, w_{k+1}), \dots, (v_s, w_s)$$

be the distinguished points of $\mathcal{L}'$.

Then there is an isomorphism between $K[L]/I_t(L)[x_{v_k,v_k}^{-1}]$ and

$$K[L']/I_{t'}(L')[x_{v_{k-1}+1,w_k}, \dots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \dots, x_{v_k,w_{k+1}+1}].$$

Proof. Under the assumption of the lemma, $\mathcal{L}'$ is a ladder and $I_{t'}(L')$ is a symmetric mixed ladder determinantal ideal. Let

$$A = K[L][x_{v_k,w_k}^{-1}]$$

and

$$B = K[L'][x_{v_{k-1}+1,w_k}, \dots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \dots, x_{v_k,w_{k+1}+1}].$$

Define a K-algebra homomorphism

$$\begin{array}{lcl} \varphi : A & \longrightarrow & B \\ x_{i,j} & \longmapsto & \begin{cases} x_{i,j} + x_{i,w_k} x_{v_k,j} x_{v_k,w_k}^{-1} & \text{if } i \neq v_k, j \neq w_k \text{ and } (i,j) \in \mathcal{L}_{(v_k,w_k)} \\ x_{i,j} & \text{otherwise.} \end{cases} \end{array}$$

The inverse of φ is

$$\begin{array}{lcl} \psi : B & \longrightarrow & A \\ x_{i,j} & \longmapsto & \begin{cases} x_{i,j} - x_{i,w_k} x_{v_k,j} x_{v_k,w_k}^{-1} & \text{if } i \neq v_k, j \neq w_k \text{ and } (i,j) \in \mathcal{L}'_{(v_k,w_k)} \\ x_{i,j} & \text{otherwise.} \end{cases} \end{array}$$

It is easy to check that φ and ψ are inverse to each other. Since

$$\varphi(I_{t_k}(L_{(v_k,w_k)})A) = I_{t_k-1}(L'_{(v_k-1,w_k-1)})B$$

we have

$$\varphi(I_t(L)A) = I_{t'}(L')B \quad \text{hence} \quad A/I_t(L)A \cong B/I_{t'}(L')B. \qquad \square$$

Using Lemma 1.6 we can establish some properties of symmetric mixed ladder determinantal ideals.

Proposition 1.7. *Symmetric mixed ladder determinantal ideals are prime and Cohen-Macaulay.*

Proof. Let $I_t(L)$ be the symmetric mixed ladder determinantal ideal associated to the ladder $\mathcal{L}$ with distinguished points $(v_1, w_1), \ldots, (v_s, w_s)$ and $t = (t_1, \ldots, t_s)$. Let $t_{\max} = \max\{t_1, \ldots, t_s\}$. If $t_{\max} = 1$ then $I_t(L)$ is generated by indeterminates, hence it is prime and Cohen-Macaulay. Therefore assume that $t_{\max} \geq 2$ and let $\widehat{\mathcal{L}}$ be the ladder with the same lower outside corners as $\mathcal{L}$, and upper outside corners $(v_k + t_{\max} - t_k, w_k + t_{\max} - t_k)$ for $k = 1, \ldots, s$. Notice that the corners are distinct, and the inequalities of Definition 1.1 are satisfied by Remark 1.4 (3). In other words, for each $k = 2, \ldots, s$ we have

$$w_k + t_{\max} - t_k < w_{k-1} + t_{\max} - t_{k-1} \quad \text{and} \quad v_k + t_{\max} - t_k > v_{k-1} + t_{\max} - t_{k-1}.$$

Let $\widehat{L} = \{x_{ij} \in X \mid (i,j) \in \widehat{\mathcal{L}},\ i \leq j\}$ and let $M = \widehat{L} \setminus L$. Denote by $I_{t_{\max}}(\widehat{L})$ the ideal generated by the minors of size $t_{\max}$ which involve only indeterminates in $\widehat{L}$. By Lemma 1.6 there exists a subset $\{z_1, \ldots, z_m\}$ of M such that

$$K[\widehat{L}]/I_{t_{\max}}(\widehat{L})[z_1^{-1}, \ldots, z_m^{-1}] \cong K[L]/I_t(L)[M][z_1^{-1}, \ldots, z_m^{-1}].$$

The ring $K[\widehat{L}]/I_{t_{\max}}(\widehat{L})$ is a Cohen-Macaulay domain by Theorem 1.13 in [4]. Therefore $K[L]/I_t(L)[M][z_1^{-1}, \ldots, z_m^{-1}]$ is a Cohen-Macaulay domain. Since M is a set of indeterminates over the ring $K[L]/I_t(L)$ and $z_1, \ldots, z_m \in M$, then $K[L]/I_t(L)[M]$ is a Cohen-Macaulay domain. Hence $I_t(L)$ is prime and Cohen-Macaulay. □

A standard argument allows us to compute the height of symmetric mixed ladder determinantal ideals. These heights have been computed by Conca in [4] for the family of cogenerated ideals. The arguments in [4] are of a more combinatorial nature, and the height is expressed as a sum of lengths of maximal chains in some subladders. Our formula for the height is very simple. The proof is independent of the results of Conca, and it essentially follows from Lemma 1.6. We use the same notation as in Definitions 1.1 and 1.3, and Lemma 1.6. An example is given in Figure 4.

Proposition 1.8. *Let $\mathcal{L}$ be a ladder with distinguished points $(v_1, w_1), \ldots, (v_s, w_s)$ and let*

$$\mathcal{H}^+ = \{(i,j) \in \mathcal{L}^+ \mid i \leq v_{k-1} - t_{k-1} + 1 \text{ or } j \leq w_k - t_k + 1 \text{ for } k = 2, \ldots, s,$$
$$j \leq w_1 - t_1 + 1,\ i \leq v_s - t_s + 1\}.$$

Let $\mathcal{H} = \mathcal{H}^+ \cup \{(j,i) \mid (i,j) \in \mathcal{H}^+\}$. Then $\mathcal{H}$ is a symmetric ladder and $htI_t(L) = |\mathcal{H}^+|$.

Proof. Observe that by Remark 1.4 (3)

$$v_k - t_k + 1 > v_{k-1} - t_{k-1} + 1, \quad \text{and} \quad w_k - t_k + 1 < w_{k-1} - t_{k-1} + 1.$$

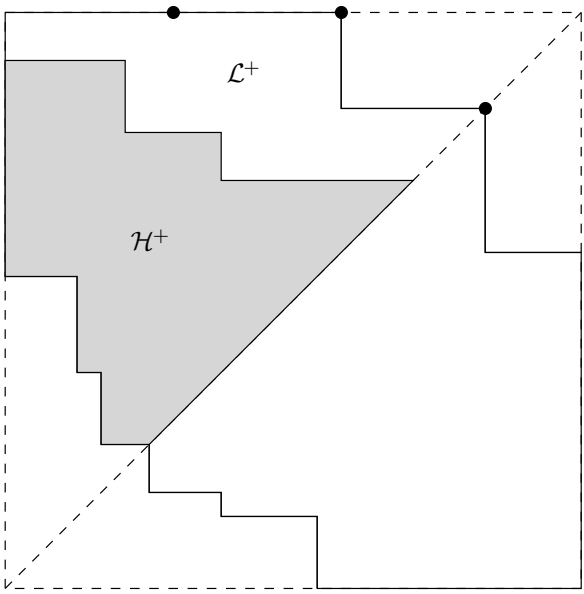

FIGURE 4. An example of $\mathcal{L}^+$ with three distinguished points and $t = (3, 6, 4)$. The corresponding $\mathcal{H}^+$ is shaded.

Therefore $\mathcal{H}$ is a ladder with upper outside corners $\{(v_k - t_k + 1, w_k - t_k + 1) \mid k = 1, \ldots, s\}$ and the same lower outside corners as $\mathcal{L}$. Let $H = \{x_{i,j} \mid (i,j) \in \mathcal{H}^+\}$. We argue by induction on $\tau = t_1 + \cdots + t_s \geq s$. If $\tau = s$, then $t_1 = \cdots = t_s = 1$, and $\mathcal{L} = \mathcal{H}$. Hence

$$I_1(L) = (x_{ij} \mid x_{ij} \in L) = I_1(H) = |\mathcal{H}^+|.$$

Assume now that the thesis holds for $\tau - 1 \geq s$ and prove it for τ. Since $\tau > s$, by Remark 1.4 (4) there exists $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$. By Lemma 1.6 we have an isomorphism between $K[L]/I_t(L)[x_{v_k,w_k}^{-1}]$ and

$$K[L']/I_{t'}(L')[x_{v_{k-1},w_k}, \ldots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \ldots, x_{v_k,w_{k+1}+1}].$$

Since x_{v_k,w_k} does not divide zero modulo $I_{t'}(L')$ and $I_t(L)$, we have

$$\text{ht } I_t(L) = \text{ht } I_{t'}(L').$$

The thesis follows by the induction hypothesis, observing that the same ladder $\mathcal{H}$ computes the height of both $I_{t'}(L')$ and $I_t(L)$. □

2. G-biliaison of symmetric mixed ladder determinantal ideals

In this section we study symmetric mixed ladder determinantal ideals from the point of view of liaison theory. We prove that they belong to the G-biliaison class of a complete intersection. In particular, they are glicci. This is yet another family of ideals of minors for which one can perform a descending G-biliaison to an ideal in the same family, in such a way that one eventually reaches an ideal generated by linear forms. Other families of ideals that were treated with an analogous technique are ideals generated by maximal minors of a matrix with polynomial entries [16], minors of a symmetric matrix with polynomial entries [11], minors of a matrix with polynomial entries [13], minors of mixed size in a ladder of a generic matrix [12], and pfaffians of mixed size in a ladder of a generic skew-symmetric matrix [9].

In [14], [15], [16] Hartshorne developed the theory of generalized divisors, which is a useful language for the study of Gorenstein liaison via the study of G-biliaison classes. In [15] it was shown that even CI-liaison and CI-biliaison generate the same equivalence classes. In [19] Kleppe, Migliore, Miró-Roig, Nagel and Peterson proved that a G-biliaison on an arithmetically Cohen-Macaulay, G_1 scheme can be realized via two G-links. The result was generalized in [16] by Hartshorne to G-biliaison on an arithmetically Cohen-Macaulay, G_0 scheme.

In Proposition 1.7 we saw that symmetric mixed ladder determinantal ideals are prime, hence they define reduced and irreducible, projective algebraic varieties. Since we wish to work in the algebraic setting, we state the definition of G-biliaison and the main theorem connecting G-biliaison and G-liaison in the language of ideals.

Definition 2.1. Let $R = K[L]$ and let $J \subseteq R$ be a homogeneous, saturated ideal. We say that J is **Gorenstein in codimension** $\leq \mathbf{c}$ if the localization $(R/J)_P$ is a Gorenstein ring for any prime ideal P of R/J of height smaller than or equal to c. We often say that J is G_c. We call **generically Gorenstein**, or G_0, an ideal J which is Gorenstein in codimension 0.

Definition 2.2. ([16], Sect. 3) Let $R = K[X]$ and let I_1 and I_2 be homogeneous ideals in R of pure height c. We say that I_1 is obtained by an **elementary G-biliaison** of height h from I_2 if there exists a Cohen-Macaulay, generically Gorenstein ideal J in R of height $c-1$ such that $J \subseteq I_1 \cap I_2$ and $I_1/J \cong [I_2/J](-h)$ as R/J-modules. If $h > 0$ we speak about **ascending** elementary G-biliaison.

The following theorem gives a connection between G-biliaison and G-liaison.

Theorem 2.3 (Kleppe, Migliore, Mirò-Roig, Nagel, Peterson [19]**; Hartshorne** [16]**).** *Let I_1 be obtained by an elementary G-biliaison from I_2. Then I_2 is G-linked to I_1 in two steps.*

We now show that symmetric mixed ladder determinantal ideals belong to the G-biliaison class of a complete intersection. The idea of the proof is as follows: starting from a symmetric mixed ladder determinantal ideal I, we construct two symmetric mixed ladder determinantal ideals I' and J such that J is contained in

$I \cap I'$ and ht I = ht I' = ht $J + 1$. We show that I can be obtained from I' by an elementary G-biliaison of height 1 on J.

Theorem 2.4. *Any symmetric mixed ladder determinantal ideal can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons.*

Proof. Let $I_t(L)$ be a symmetric mixed ladder determinantal ideal associated to a ladder $\mathcal{L}^+$ with distinguished points $(v_1, w_1), \ldots, (v_s, w_s)$. Let $L_k = L_{(v_k, w_k)}$, then

$$I_t(L) = I_{t_1}(L_1) + \cdots + I_{t_s}(L_s) \subseteq K[L].$$

As discussed in Remark 1.4 (1) we will not distinguish between symmetric mixed ladder determinantal ideals and their extensions. Therefore, all ideals will be in $R = K[L]$. If $t_1 = \cdots = t_s = 1$ then $I_t(L)$ is generated by linear forms. Hence let $t_k = \max\{t_1, \ldots, t_s\} \geq 2$. From Remark 1.4 (3) we have that $w_{k+1} - w_k < 0 < v_k - v_{k-1}$. In particular (v_k, w_k) is an upper outside corner.

Let $\mathcal{L}'^+$ be the ladder with distinguished points

$$(v_1, w_1), \ldots, (v_{k-1}, w_{k-1}), (v_k - 1, w_k - 1), (v_{k+1}, w_{k+1}), \ldots, (v_s, w_s).$$

Observe that $\mathcal{L}'^+$ is obtained from $\mathcal{L}^+$ by removing the entries

$$(v_{k-1} + 1, w_k), \ldots, (v_k - 1, w_k), (v_k, w_k), (v_k, w_k + 1), \ldots, (v_k, w_{k+1} - 1).$$

Let $t' = (t_1, \ldots, t_{k-1}, t_k - 1, t_{k+1}, \ldots, t_s)$, and let $I_{t'}(L')$ be the associated symmetric mixed ladder determinantal ideal. It is easy to check that $\mathcal{L}'^+$ and t' satisfy the inequalities of Definition 1.3 and of Remarks 1.4. By Proposition 1.8

$$\text{ht } I_t(L) = \text{ht } I_{t'}(L') = |\mathcal{H}^+|$$

where

$$\mathcal{H}^+ = \{(i,j) \in \mathcal{L}^+ \mid i \leq v_{k-1} - t_{k-1} + 1 \text{ or } j \leq w_k - t_k + 1 \text{ for } k = 2, \ldots, s,$$
$$j \leq w_1 - t_1 + 1,\ i \leq v_s - t_s + 1\}.$$

Let $\mathcal{J}^+$ be the ladder obtained from $\mathcal{L}^+$ by removing (v_k, w_k), and let

$$(v_1, w_1), \ldots, (v_{k-1}, w_{k-1}), (v_k - 1, w_k), (v_k, w_k - 1), (v_{k+1}, w_{k+1}), \ldots, (v_s, w_s)$$

be its distinguished points (see Figure 5). Let $u = (t_1, \ldots, t_{k-1}, t_k, t_k, t_{k+1}, \ldots, t_s)$. Then

$$I_u(J) = I_{t_1}(L_1) + \cdots + I_{t_{k-1}}(L_{k-1}) + I_{t_k}(J_{(v_k - 1, w_k)}) + I_{t_k}(J_{(v_k, w_k - 1)})$$
$$+ I_{t_{k+1}}(L_{k+1}) + \cdots + I_{t_s}(L_s).$$

In other words, $I_u(J)$ is the ideal generated by the minors of $I_t(L)$ that do not involve the indeterminate x_{v_k, w_k}. We claim that $I_u(J) \subseteq I_t(L) \cap I_{t'}(L')$. It is clear that $I_u(J) \subseteq I_t(L)$. The inclusion $I_u(J) \subseteq I_{t'}(L')$ follows from $I_{t_k}(L_{(v_k - 1, w_k)}) + I_{t_k}(L_{(v_k, w_k - 1)}) \subset I_{t_k - 1}(L'_{(v_k - 1, w_k - 1)})$.

Let

$$\mathcal{I}^+ = \mathcal{H}^+ \setminus \{(v_k - t_k + 1, w_k - t_k + 1)\}.$$

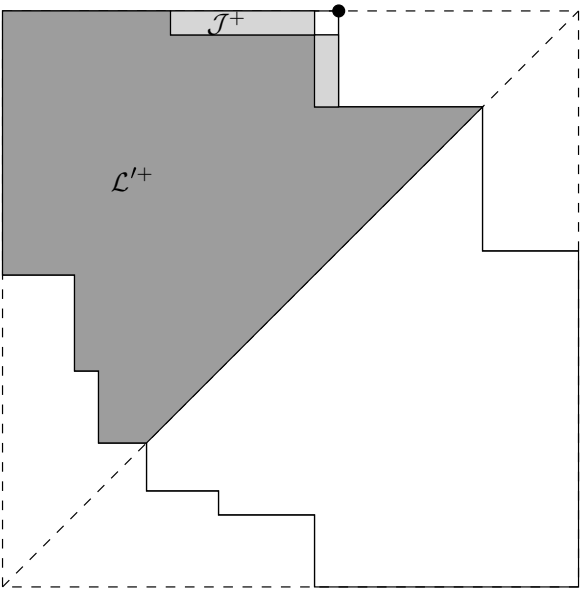

FIGURE 5. An example of $\mathcal{L}^+$ with $\mathcal{L}'^+$ and $\mathcal{J}^+$. The distinguished point (v_k, w_k) is marked. $\mathcal{L}'^+$ is colored in a darker shade and the entries which belong to $\mathcal{J}^+$ but not to $\mathcal{L}'^+$ are colored in a lighter shade.

By Proposition 1.8

$$\text{ht } I_u(J) = |\mathcal{I}^+| = \text{ht } I_t(L) - 1.$$

The ideal $I_u(J)$ is prime and Cohen-Macaulay by Proposition 1.7. In particular it is generically Gorenstein.

We claim that $I_t(L)$ is obtained from $I_{t'}(L')$ by an elementary G-biliaison of height 1 on $I_u(J)$. This is equivalent to showing that

$$I_t(L)/I_u(J) \cong [I_{t'}(L')/I_u(J)](-1) \tag{1}$$

as $R/I_u(J)$-modules. Denote by $[\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_t]$ the $t \times t$-minor of X which involves rows $\alpha_1, \ldots, \alpha_t$ and columns $\beta_1, \ldots, \beta_t$. We claim that multiplication by

$$f = \frac{[v_k - t_k + 1, \ldots, v_k - 1; w_k - t_k + 1, \ldots, w_k - 1]}{[v_k - t_k + 1, \ldots, v_k - 1, v_k; w_k - t_k + 1, \ldots, w_k - 1, w_k]}$$

yields an isomorphism between $I_t(L)/I_u(J)$ and $[I_{t'}(L')/I_u(J)](-1)$.

Notice in fact that the ideal $I_t(L)/I_u(J)$ is generated by the minors of size $t_k \times t_k$ of L_k which involve both row v_k and column w_k, while the ideal $I_{t'}(L')/I_u(J)$ is generated by the minors of size $(t_k - 1) \times (t_k - 1)$ of L'_k. For any minor $[\alpha_1, \ldots, \alpha_{t_k-1}, v_k; \beta_1, \ldots, \beta_{t_k-1}, w_k] \in I_{t_k}(L_k)$ which involves both row v_k and column w_k, consider the minor $[\alpha_1, \ldots, \alpha_{t_k-1}; \beta_1, \ldots, \beta_{t_k-1}] \in I_{t_k-1}(L'_k)$.

By [11], Lemma 2.6

$$[\alpha_1,\dots,\alpha_{t_k-1};\beta_1,\dots,\beta_{t_k-1}]\cdot[v_k-t_k+1,\dots,v_k-1,v_k;w_k-t_k+1,\dots,w_k-1,w_k]=$$
$$[v_k-t_k+1,\dots,v_k-1;w_k-t_k+1,\dots,w_k-1]\cdot[\alpha_1,\dots,\alpha_{t_k-1},v_k;\beta_1,\dots,\beta_{t_k-1},w_k]$$

modulo $I_u(J)$. Therefore the ideals

$$[v_k-t_k+1,\dots,v_k-1;w_k-t_k+1,\dots,w_k-1]\cdot I_t(L)+I_u(J)$$

and

$$[v_k-t_k+1,\dots,v_k;w_k-t_k+1,\dots,w_k]\cdot I_{t'}(L')+I_u(J)$$

are equal, hence they are equal modulo $I_u(J)$. Therefore isomorphism (1) holds, and $I_t(L)$ and $I_{t'}(L')$ are G-bilinked on $I_u(J)$. Repeating this procedure, one eventually reaches the ideal generated by the entries of the ladder $\mathcal{H}$ defined in Proposition 1.8. Clearly

$$I_1(H)=(x_{ij}\mid(i,j)\in\mathcal{H})$$

is a complete intersection. □

The following is a straightforward consequence of Theorem 2.4, according to Theorem 2.3.

Corollary 2.5. *Every symmetric mixed ladder determinantal ideal $I_t(L)$ can be G-linked in $2(t_1+\cdots+t_s)$ steps to a complete intersection of linear forms of the same height. Hence symmetric mixed ladder determinantal ideals are glicci.*

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Liaison Invariants and the Hilbert Scheme of Codimension 2 Subschemes in $^{n+2}$

Jan O. Kleppe

Abstract. In this paper we study the Hilbert scheme $\mathrm{Hilb}^{p(v)}(\ \ )$ of equidimensional locally Cohen-Macaulay codimension 2 subschemes, with a special look to surfaces in ^{4} and 3-folds in ^{5}, and the Hilbert scheme stratification $\mathrm{H}_{\gamma,\rho}$ of constant cohomology. For every $(X) \in \mathrm{Hilb}^{p(v)}(\ \ )$ we define a number δ_X in terms of the graded Betti numbers of the homogeneous ideal of X and we prove that $1+\delta_X-\dim_{(X)}\mathrm{H}_{\gamma,\rho}$ and $1+\delta_X-\dim T_{\gamma,\rho}$ are CI-biliaison invariants where $T_{\gamma,\rho}$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X). As a corollary we get a formula for the dimension of any generically smooth component of $\mathrm{Hilb}^{p(v)}(\ \ )$ in terms of δ_X and the CI-biliaison invariant. Both invariants are equal in this case.

Recall that, for space curves C, Martin-Deschamps and Perrin have proved the smoothness of the "morphism" $\phi : \mathrm{H}_{\gamma,\rho} \to E_\rho :=$ isomorphism classes of graded modules M satisfying $\dim M_v = \rho(v)$, given by sending C onto its Rao module. For surfaces X in ^{4} we have two Rao modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ of dimension $\rho_i(v)$, $\rho := (\rho_1, \rho_2)$ and an induced extension $b \in {}_0\mathrm{Ext}^2(M_2, M_1)$ and a result of Horrocks and Rao saying that a triple $D := (M_1, M_2, b)$ of modules M_i of finite length and an extension b as above determine a surface X up to biliaison. We prove that the corresponding "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho =$ isomorphism classes of graded modules M_i satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, is smooth, and we get a smoothness criterion for $\mathrm{H}_{\gamma,\rho}$, i.e., for the equality of the two biliaison invariants. Moreover we get some smoothness results for $\mathrm{Hilb}^{p(v)}(\ \ )$, valid also for 3-folds, and we give examples of obstructed surfaces and 3-folds. The linkage result we prove in this paper turns out to be useful in determining the structure and dimension of $\mathrm{H}_{\gamma,\rho}$, and for proving the main biliaison theorem above.

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1. Introduction

A main object of this paper is to find the dimension of the Hilbert scheme, $\mathrm{Hilb}^{p(v)}(\mathbb{P})$, of equidimensional locally Cohen-Macaulay (lCM) codimension 2 subschemes of $\mathbb{P} := \mathbb{P}^{n+2}$. As an initial ambitious goal we look for a formula for the dimension of any reduced component V of the Hilbert scheme $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ in terms of the graded Betti numbers of the homogeneous ideal I_X of a general element (X) of V. Since we expect the matrices in the minimal resolution of I_X to play a role, it seems natural to modify our goal by introducing a biliaison invariant in the dimension formula. Indeed in this paper we explicitly define an invariant $\delta_X^{n+1}(-n-3)$ in terms of the graded Betti numbers of I_X and $H^n_*(\mathcal{O}_X)$ and we prove that

$$\dim V = 1 + \delta_X^{n+1}(-n-3) - \mathrm{sumext}(X)$$

where $\mathrm{sumext}(X)$ is a CI-biliaison invariant (Corollary 9.4). In the case X is a curve ($n = 1$) with Hartshorne-Rao module M, we use results of [38] to prove

$$\mathrm{sumext}(X) = \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M, M) \ ,$$

(Theorem 3.7 and Remark 3.9) and there is a similar, but much more complicated formula in the surface case (which we may deduce from Remark 6.4).

Let $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P})$ be the Hilbert scheme whose k-points (X) corresponds to equidimensional lCM codimension 2 subschemes X of $\mathbb{P}^{n+2}$ with constant cohomology (see [38] for the curve case). If X is *any* equidimensional lCM codimension 2 subscheme of $\mathbb{P}$, we define $\mathrm{obsumext}(X)$ in the following way,

$$\mathrm{obsumext}(X) = 1 + \delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} \, .$$

We define $\mathrm{sumext}(X)$ by the same expression provided we have replaced $\mathrm{H}_{\gamma,\rho}$ by its tangent space, $T_{\gamma,\rho}$, at (X). Then we prove that $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ are CI-biliaison invariants (Theorem 9.1). Since every arithmetically Cohen-Macaulay codimension 2 subscheme is in the liaison class of a complete intersection (CI) by Gaeta's theorem, it follows that $\mathrm{sumext}(X) = \mathrm{obsumext}(X) = 0$ and that $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}) = 1 + \delta_X^{n+1}(-n-3)$ for $n > 0$ if X is arithmetically Cohen-Macaulay (Corollary 9.6). Even though we do not prove the explicit expression of $\mathrm{sumext}(X)$ in terms of the Rao modules of X in general, the theorem is motivated from the fact that the Rao modules are invariant under biliaison up to shift. In fact it seems more effective to compute $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ by considering a nice representative X' in its even liaison class, e.g., the minimal element, and to compute $\delta_{X'}^{n+1}(-n-3)$, $\dim_{(X')} \mathrm{H}_{\gamma,\rho}$, and $\dim T_{\gamma,\rho}$ for X'.

Since the curve case of the results above is rather well understood ([38], [33]), we will in the present paper mostly concentrate on the study of the Hilbert scheme $\mathrm{H}(d, p, \pi)$ of surfaces of degree d and arithmetic (resp. sectional) genus p (resp. π). Recall that, for space curves C, Martin-Deschamps and Perrin proved the smoothness of the "morphism" $\phi : \mathrm{H}_{\gamma,\rho} \to E_\rho$: = isomorphism classes of graded R-modules M satisfying $\dim M_v = \rho(v)$, given by sending C onto its Rao module.

Earlier Rao proved that any graded R-module M of finite length determines the liaison class of a curve, up to dual and shift in the grading ([47]). Note that Rao's result is related to the surjectivity of ϕ, while the smoothness of ϕ implies infinitesimal surjectivity. For surfaces in $\mathbb{P}^4$ there is a result in Bolondi's paper [6], stating that a triple $D := (M_1, M_2, b)$ of graded modules M_i of finite length and an extension $b \in {}_0\mathrm{Ext}^2(M_2, M_1)$ determine the biliaison class of a surface X such that $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ modulo some shift in the grading. The result is a consequence of the main theorem of [48] and Horrocks' classification of stable vector bundles ([24]), as mentioned by Rao in [48]. Therefore it is natural to consider the stratification $\mathrm{H}_{\gamma,\rho}$ of $\mathrm{H}(d, p, \pi)$ where now $\rho := (\rho_1, \rho_2)$ and $\rho_i(v) = \dim H^i(\mathcal{I}_X(v))$, and to ask for the smoothness of the corresponding "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho :=$ isomorphism classes of triples (M_1, M_2, b) where M_i are graded R-modules which satisfy $\dim(M_i)_v = \rho_i(v)$ and where an isomorphism between triples is an isomorphism between the corresponding modules which commutes with the extensions. We prove in Section 5 that the answer is yes (Theorem 5.3). As a corollary we get a smoothness criterion for $\mathrm{H}_{\gamma,\rho}$ (Corollary 5.4, Remark 6.3), i.e., for the equality $\mathrm{sumext}(X) = \mathrm{obsumext}(X)$ to hold. Note that we do not prove that φ extends to a morphism of schemes; we only prove that the corresponding morphism of the local deformation functors is formally smooth. This, however, takes fully care of what we want.

In Section 6 we determine the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X), and we prove a local isomorphism $\mathrm{H}_{\gamma,\rho} \simeq \mathrm{H}(d, p, \pi)$ at (X) under some conditions (Proposition 6.1, Remark 6.2). Note, however, that if X has seminatural cohomology, we know that $\mathrm{H}_{\gamma,\rho} \simeq \mathrm{H}(d, p, \pi)$ at (X) by the semicontinuity of $\dim H^i(\mathcal{I}_X(v))$ and this observation mostly suffices for our applications. In Section 7 we prove a useful linkage result (Theorem 7.1) which we apply to determine the structure and the dimension of $\mathrm{H}_{\gamma,\rho}$ and to prove our main theorem on the biliaison invariants. In this section we also give conditions for a linked surface to be, e.g., non-generic, thus proving the existence of surfaces with "smaller" cohomology in some cases (Proposition 7.4).

Since the technical problems in describing well the stratification of $\mathrm{H}(d, p, \pi)$ and the morphism φ are quite complicated (see [31]), we don't follow up this trace for equidimensional lCM codimension 2 subschemes $X \subseteq \mathbb{P}^{n+2}$ of dimension $n \geq 3$. Instead we only use our main theorem on the biliaison invariance of $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ together with some new results on the smoothness and the dimension of $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ in our study of the Hilbert schemes of, e.g., 3-folds in Section 9. We also give a vanishing criterion for $H^1(\mathcal{N}_X)$, but unfortunately, as in [33], the results we get require that the Hartshorne-Rao modules are rather "small". When the conditions of these vanishing criteria do not hold, we give examples of obstructed surfaces and 3-folds.

2. Notations and terminology

A surface (resp. curve) X is an equidimensional, locally Cohen-Macaulay subscheme (lCM) of $\mathbb{P}^4$ (resp. $\mathbb{P}^3$) of dimension 2 (resp. 1) with sheaf ideal $\mathcal{I}_X$ and normal sheaf $\mathcal{N}_X = \mathrm{Hom}_{\mathcal{O}}\ (\mathcal{I}_X, \mathcal{O}_X)$. If $\mathcal{F}$ is a coherent $\mathcal{O}$ -Module, we let $H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$, $H^i_*(\mathcal{F}) = \oplus_v H^i(\mathcal{F}(v))$ and $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$, and we denote by $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$ the Euler-Poincaré characteristic. Then $p(v) = \chi(\mathcal{O}_X(v))$ is the Hilbert polynomial of X. Put $n = \dim X$ and

$$s(X) = \min\{v | h^0(\mathcal{I}_X(v)) \neq 0\},$$
$$e(X) = \max\{v | h^n(\mathcal{O}_X(v)) \neq 0\}.$$

Let $I = I_X = H^0_*(\mathcal{I}_X)$ be the homogeneous ideal. I is a graded module over the polynomial ring $R = k[X_0, X_1, \ldots, X_{n+2}]$, where k is supposed to be algebraically closed (and of characteristic zero in Sections 5, 6 and in all examples since we there may use results and methods of papers relying on this assumption). The postulation γ of X is the function defined over the integers by $\gamma(v) = \gamma_X(v) = h^0(\mathcal{I}_X(v))$.

Let $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ denote the Hilbert scheme of equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ with Hilbert polynomial p (cf. [21]). X is called *unobstructed* if $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ is smooth at the corresponding point (X), otherwise X is obstructed. A subscheme of $\mathbb{P}^{n+2}$ belonging to a sufficiently small open irreducible subset U of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ (small enough so that any (X) of U satisfies all the openness properties which we want it to have) is called a *generic* subscheme of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$, and accordingly, if we state that a generic subscheme has a certain property, then there is a non-empty open irreducible subset of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ of subschemes having this property.

In the case of curves we put $\mathrm{H}(d, g) = \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ provided $p(v) = dv + 1 - g$. Moreover we let $M = M(C) := H^1_*(\mathcal{I}_C)$ be the deficiency or Hartshorne-Rao module of the curve C. The deficiency function ρ is defined by $\rho(v) = h^1(\mathcal{I}_C(v))$. Let $\mathrm{H}(d, g)_{\gamma,\rho}$ (resp. $\mathrm{H}(d, g)_\gamma$) denote the subscheme of $\mathrm{H}(d, g)$ of curves with constant cohomology given by γ and ρ, (resp. constant postulation γ), where "constant" means flat deformations of the corresponding modules, see [38]. Let Def_M be the local deformation functor consisting of graded deformations M_S of M to $\mathbb{P}^3 \times \mathrm{Spec}(S)$ modulo graded isomorphisms of M_S over M, where S is a local artinian k-algebra with residue field k, i.e., such that M_S is S-flat and $M_S \otimes k = M$.

For a surface X we define the arithmetic genus p by $p = \chi(\mathcal{O}_X) - 1$, while the sectional genus π is given by $\chi(\mathcal{O}_X(1)) = d - \pi + 1 + \chi(\mathcal{O}_X)$. By Riemann-Roch's theorem we have

$$p(v) = \chi(\mathcal{O}_X(v)) = \frac{1}{2}dv^2 - \left(\pi - 1 - \frac{1}{2}d\right)v + \chi(\mathcal{O}_X). \tag{1}$$

Put $\mathrm{H}(d, p, \pi) = \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ in this case. Moreover let $M_i = M_i(X)$ be the deficiency modules $H^i_*(\mathcal{I}_X)$ for i = 1,2. The deficiency $\rho = (\rho_1, \rho_2)$ of X is the function defined over the integers by $\rho(v) = \rho_X(v) = (\rho_1(v), \rho_2(v))$ where $\rho_i(v) = h^i(\mathcal{I}_X(v))$ for $i = 1, 2$. Let $\mathrm{H}_{\gamma,\rho} = \mathrm{H}(d, p, \pi)_{\gamma,\rho}$ (resp. $\mathrm{H}_\gamma = \mathrm{H}(d, p, \pi)_\gamma$) denote the

subscheme of $\mathrm{H}(d,p,\pi)$ of surfaces with constant cohomology given by γ and ρ, (resp. constant postulation γ) where again "constant" means flat deformations of the corresponding modules.

For the notion of linkage, we refer to [39]. Note that liaison (resp. even liaison or biliaison) is the equivalence relation generated by linkage (resp. direct linkages in an even number of steps).

For any graded R-module N, we have the right derived functors $H^i_{\mathfrak{m}}(N)$ and ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N,-)$ of $\Gamma_{\mathfrak{m}}(N) := \oplus_v \ker(N_v \to \Gamma(\mathbb{P}, \tilde{N}(v)))$ and $\Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(N,-))_v$ respectively (cf. [20], exp. VI or [22]) where $\mathfrak{m} = (X_0, \ldots, X_{n+2})$. We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, e.g., ${}_v\mathrm{ext}^i_{\mathfrak{m}}(N_1,N_2) = \dim {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2)$.

Let N_1 and N_2 be graded R-modules of finite type. As in [33] we need the spectral sequence

$$E_2^{p,q} = {}_v\mathrm{Ext}^p_R(N_1, H^q_{\mathfrak{m}}(N_2)) \Rightarrow {}_v\mathrm{Ext}^{p+q}_{\mathfrak{m}}(N_1,N_2) \tag{2}$$

([20], exp. VI) and the duality isomorphism

$${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_2,N_1) \cong {}_{-v-n-3}\mathrm{Ext}^{n+3-i}_R(N_1,N_2)^\vee, \quad i,v \in \mathbb{Z} \tag{3}$$

where $(-)^\vee = \mathrm{Hom}_k(-,k)$ (cf. [30], Thm. 1.1, see [28], Thm. 2.1.4 for a full proof). Moreover there is a long exact sequence

$$\to {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2) \to {}_v\mathrm{Ext}^i_R(N_1,N_2) \to \mathrm{Ext}^i_{\mathcal{O}}\ (\tilde{N}_1, \tilde{N}_2(v)) \to {}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_1,N_2) \to \tag{4}$$

([20], exp. VI) which at least for equidimensional, lCM subschemes of codimension 2 (with $n > 0$) relate the deformation theory of X, described by $H^{i-1}(\mathcal{N}_X) \simeq \mathrm{Ext}^i_{\mathcal{O}}\ (\tilde{I}, \tilde{I})$ for $i = 1, 2$ (cf. [28], Rem. 2.2.6), to the deformation theory of the homogeneous ideal $I = I_X$, described by ${}_0\mathrm{Ext}^i_R(I,I)$, in the following exact sequence

$$\begin{aligned} 0 \to {}_v\mathrm{Ext}^1_R(I,I) \to H^0(\mathcal{N}_X(v)) \to {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \\ \xrightarrow{\alpha} {}_v\mathrm{Ext}^2_R(I,I) \to H^1(\mathcal{N}_X(v)) \to {}_v\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to \end{aligned} \tag{5}$$

see [49] or [17] for related works on such deformation functors.

3. The dimension of $\mathrm{H}(d,g)$ and biliaison invariants

In this section we consider the Hilbert scheme, $\mathrm{H}(d,g)$, of curves in $\mathbb{P}^3$ and results which we would like to generalize to surfaces in $\mathbb{P}^4$. We will focus on the dimension of the Hilbert schemes and some biliaison invariants which we naturally detect from this point of view.

Recall that $\chi(\mathcal{N}_C(v)) = 2dv + 4d$ and that $\chi(\mathcal{N}_C) = 4d$ is a lower bound for $\dim_{(C)} \mathrm{H}(d,g)$. For this reason the number $4d$ is often called the expected dimension of $\mathrm{H}(d,g)$ even though it often does not give the correct dimension of $\mathrm{H}(d,g)$ at (C). E.g., at ACM, generically complete intersection curves the dimension is never $4d$ if $e(C) \geq s(C)$.

To give a more reliable estimate for the dimension of the components of $\mathrm{H}(d,g)$, we have found it convenient to introduce the following invariant, defined in terms of the numbers $n_{j,i}$ appearing in a minimal resolution of the homogeneous ideal I_C of C:

$$0 \to \bigoplus_{i=1}^{r_3} R(-n_{3,i}) \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I_C \to 0\ . \tag{6}$$

Note that we can define the graded Betti numbers, $\beta_{j,k}$, of I_C by just putting $\oplus_{k=1}^{\infty} R(-k)^{\beta_{j,k}} := \oplus_{i=1}^{r_j} R(-n_{j,i})$.

Definition 3.1. If C is a curve in $\mathbb{P}^3$, we let

$$\delta_C^j(v) := \sum_i h^j(\mathcal{I}_C(n_{1,i}+v)) - \sum_i h^j(\mathcal{I}_C(n_{2,i}+v)) + \sum_i h^j(\mathcal{I}_C(n_{3,i}+v)).$$

Put $\delta^j(v) = \delta_C^j(v)$. In [33] we proved the following result (Lem. 2.2 of [33]).

Lemma 3.2. *Let C be any curve of degree d in $\mathbb{P}^3$. Then the following expressions are equal*

$${}_0\mathrm{ext}_R^1(I_C,I_C) - {}_0\mathrm{ext}_R^2(I_C,I_C) = 1-\delta^0(0) = 4d+\delta^2(0)-\delta^1(0) = 1+\delta^2(-4)-\delta^1(-4).$$

Remark 3.3. Comparing with the results and notations of [38] we recognize $1-\delta^0(0)$ as δ_γ and $\delta^1(-4)$ as $\epsilon_{\gamma,\delta}$ in their terminology. By Lemma 3.2 it follows that the dimension of the Hilbert scheme $\mathrm{H}_{\gamma,M}$ of constant postulation and Rao module, which they show is $\delta_\gamma + \epsilon_{\gamma,\delta} - {}_0\mathrm{hom}(M,M)$ (Thm. 3.8, page 171), is also equal to $1+\delta^2(-4) - {}_0\mathrm{hom}(M,M)$.

Note that the difference of the ext-numbers in Lemma 3.2 is a lower bound for $\dim O_{\mathrm{H}(d,g)_\gamma,(C)}$ ([33], proof of Thm. 2.6 (i)). Mainly since $\mathrm{H}(d,g)_\gamma$ is a subscheme of $\mathrm{H}(d,g)$, we used this lower bound in [35], Thm. 24, to prove the following result

Theorem 3.4. *Let C be a curve in $\mathbb{P}^3$ and let $\delta^j(v) = \delta_C^j(v)$ for any j and v. Then the dimension of $\mathrm{H}(d,g)$ at (C) satisfies*

$$\dim_{(C)} \mathrm{H}(d,g) \geq 1-\delta^0(0) = 4d+\delta^2(0)-\delta^1(0).$$

Moreover if C is a generic curve of a generically smooth component V of $\mathrm{H}(d,g)$ and $M = H^1_*(\mathcal{I}_C)$, then

$$\dim V = 4d+\delta^2(0)-\delta^1(0) + {}_{-4}\mathrm{hom}_R(I_C,M)$$

where ${}_{-4}\mathrm{Hom}_R(I_C,M)$ is the kernel of the map

$$\bigoplus_i H^1(\mathcal{I}_C(n_{1,i}-4)) \to \bigoplus_i H^1(\mathcal{I}_C(n_{2,i}-4))$$

induced by the corresponding map in (6).

Remark 3.5. Let C be any curve in $\mathbb{P}^3$ and suppose

$${}_{-4}\mathrm{Hom}_R(I_C,M) = {}_0\mathrm{Hom}_R(I_C,M) = 0.$$

Then C is unobstructed and the lower bound of the inequality of Theorem 3.4 is equal to $\dim_{(C)} \mathrm{H}(d,g)$ by Thm. 2.6 of [33].

Remark 3.6. Let C be any curve in $\mathbb{P}^3$.

(i) If $M = 0$, then $\delta^1(0) = 0$ and we can use Remark 3.5 to see that C is unobstructed and that the lower bound of Theorem 3.4 is equal to $\dim_{(C)} \mathrm{H}(d, g)$. This coincides with [13].

(ii) If $\operatorname{diam} M = 1$, $\dim M = r$ and C is a generic curve, then C is unobstructed by [33] Cor. 1.6 and the lower bound is equal to $4d + \delta^2(0) + r\beta_{2,c}$. Indeed $r\beta_{1,c} = 0$ for a generic curve by [33], Cor. 4.4. Moreover in this case the "correction" number ${}_{-4}\hom_R(I_C, M)$ is equal to $r\beta_{1,c+4}$. Hence we get

$$\dim V = 4d + \delta^2(0) + r(\beta_{2,c} + \beta_{1,c+4}).$$

This coincides with the dimension formula of [33], Thm. 3.4.

Theorem 3.4 is a consequence of the inclusion $\mathrm{H}(d, g)_\gamma \hookrightarrow \mathrm{H}(d, g)$ of schemes. One may try the same argument for the inclusion $\mathrm{H}(d, g)_{\gamma,\rho} \hookrightarrow \mathrm{H}(d, g)$ since we also for these schemes know tangent and obstruction spaces. This leads to

Theorem 3.7. *Let C be a curve in $\mathbb{P}^3$ and $M = H^1_*(\mathcal{I}_C)$. Then the dimension of $\mathrm{H}(d, g)$ at (C) satisfies*

$$\dim_{(C)} \mathrm{H}(d, g) \geq 1 + \delta^2(-4) - \sum\nolimits_{i=0}^{2} {}_0\mathrm{ext}^i_R(M, M).$$

Moreover if C is a generic curve of a generically smooth component V of $\mathrm{H}(d, g)$, then

$$\begin{aligned}\dim V &= 4d + \delta^2(0) - \delta^1(0) + \delta^1(-4) - \sum\nolimits_{i=0}^{1} {}_0\mathrm{ext}^i_R(M, M)\\ &= 1 + \delta^2(-4) - \sum\nolimits_{i=0}^{1} {}_0\mathrm{ext}^i_R(M, M).\end{aligned}$$

Proof. We consider the stratification $\mathrm{H}(d, g)_{\gamma,\rho}$ of the Hilbert scheme $\mathrm{H}(d, g)$ and the "morphism" $\phi : \mathrm{H}(d, g)_{\gamma,\rho} \to E_\rho :=$ isomorphism classes of R-modules M given by mapping (C) onto $M(C)$. By [38], Thm. 1.5, ϕ is smooth, and $\mathrm{H}(d, g)_{\gamma,M} := \phi^{-1}(M)$ is a scheme of dimension $1 + \delta^2(-4) - {}_0\hom(M, M)$ (see Remark 3.3). If we ignore the scheme structures, we may still, for each curve C, consider the corresponding local deformation functor, ϕ_C, of ϕ at (C), defined on the category of local artinian k-algebras with residue field k. ϕ_C is smooth of fiber dimension as above by the results of [38], see also [33], Rem. 2.12 for the curve case and Theorem 5.3 of this paper for the corresponding result for surfaces.

It is well known that ${}_0\mathrm{Ext}^i_R(M, M)$ for $i = 1, 2$, determine the local graded deformation functor, Def_M, of the R-module $M := M(C)$, e.g.,

$${}_0\mathrm{ext}^1(M, M) - {}_0\mathrm{ext}^2(M, M) \leq \dim E_{\rho,M} \leq {}_0\mathrm{ext}^1(M, M),$$

where $E_{\rho,M}$ is the hull of Def_M ([37], Thm. 4.2.4). Moreover we have equality to the right if and only if Def_M is formally smooth. Combining with the smoothness of ϕ_C and its fiber dimension we get

$$\begin{aligned}1 + \delta^2(-4) - \sum_{i=0}^{2} {}_0\mathrm{ext}^i(M, M) &\leq \dim_{(C)} \mathrm{H}(d, g)_{\gamma,\rho}\\ &\leq 1 + \delta^2(-4) - {}_0\hom(M, M) + {}_0\mathrm{ext}^1(M, M)\end{aligned} \tag{7}$$

with equality to the right if and only if $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C). This proves the inequality of the theorem since $\dim_{(C)}\mathrm{H}(d,g) \geq \dim_{(C)}\mathrm{H}(d,g)_{\gamma,\rho}$. We also get the final statement because, at a generic curve C with postulation γ and deficiency ρ, $\mathrm{H}(d,g)_{\gamma,\rho} \cong \mathrm{H}(d,g)$ around (C)! Indeed if we have $\dim_{(C)}\mathrm{H}(d,g)_{\gamma,\rho} < \dim_{(C)}\mathrm{H}(d,g)$, then a small neighborhood of (C) in $\mathrm{H}(d,g)_{\gamma,\rho}$ is not *open* in $\mathrm{H}(d,g)$, contradicting the assumption that C is generic in $\mathrm{H}(d,g)$. Hence we have equality in dimensions and in fact a local isomorphism (e.g., by generic flatness) since $\mathrm{H}(d,g)$ is smooth at (C). It follows that $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C) and the inequality of (7) to the right is an equality. □

Remark 3.8. Let $T_{\gamma,\rho}$ be the tangent space of $\mathrm{H}(d,g)_{\gamma,\rho}$ at (C). Then we easily see from the proof that the upper bound in (7) is equal to $\dim T_{\gamma,\rho}$.

If we want to generalize Theorem 3.7 to codimension 2 subschemes in $\mathbb{P}^{n+2}$, the explicit replacements of $\sum_{i=0}^{1} {}_0\mathrm{ext}^i(M,M)$ in the generalized statements seem to be very complicated. However observing that $\sum_{i=0}^{1} {}_0\mathrm{ext}^i(M,M)$ is a biliaison invariant (since M is, up to a twist), it seems to be the following weaker form of Theorem 3.7 and (7) which is natural to generalize:

Remark 3.9. If we define $\mathrm{sumext}(C)$ and $\mathrm{obsumext}(C)$ by $\mathrm{sumext}(C) = 1 + \delta^2(-4) - \dim T_{\gamma,\rho}$ and $\mathrm{obsumext}(C) = 1 + \delta^2(-4) - \dim_{(C)}\mathrm{H}(d,g)_{\gamma,\rho}$, then $\mathrm{sumext}(C)$ and $\mathrm{obsumext}(C)$ are biliaison invariants. We have

$$\mathrm{sumext}(C) \leq \mathrm{obsumext}(C)$$

and equality holds if and only if $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C). Furthermore if C is unobstructed and generic in $\mathrm{H}(d,g)$, then

$$\dim_{(C)}\mathrm{H}(d,g) = 1 + \delta^2(-4) - \mathrm{sumext}(C).$$

We have not yet proved that $\mathrm{obsumext}(C)$ is a biliaison invariant, but it will follow from later results, or from [38], Thm. 1.5 and Remark 3.3.

For curves we have

$$\mathrm{sumext}(C) = \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M,M) \tag{8}$$

and

$$\sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M,M) \leq \mathrm{obsumext}(C) \leq \sum_{i=0}^{2} {}_0\mathrm{ext}^i_R(M,M) \tag{9}$$

which we may use to compute $\mathrm{sumext}(C)$ and estimate $\mathrm{obsumext}(C)$. We may also compute these invariants somewhere in the even liaison class, e.g., by letting C be the minimal curve and computing $\dim_{(C)}\mathrm{H}(d,g)_{\gamma,\rho}$, $\dim T_{\gamma,\rho}$ and $\delta^2(-4)$ in this case. If D is in the even liaison class of C, $D \in \mathrm{H}_{\gamma',\rho'}$, and if we can compute $\delta^2_D(-4)$, then we get the dimensions of $\mathrm{H}_{\gamma',\rho'}$ and $T_{\gamma',\rho'}$, from the biliaison invariants.

4. The dimension and the smoothness of $\mathrm{H}(d, p, \pi)$

In this section we consider the Hilbert scheme, $\mathrm{H}(d, p, \pi)$, of surfaces in $\mathbb{P}^4$. Our goal is to see how far we can generalize the results of the preceding section to surfaces. We will focus on the dimension and the smoothness of the Hilbert scheme.

To compute the dimension of the components of $\mathrm{H}(d, p, \pi)$, we consider the minimal resolution of $I = I_X$:

$$0 \to \bigoplus_{i=1}^{r_4} R(-n_{4,i}) \to \bigoplus_{i=1}^{r_3} R(-n_{3,i}) \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0, \tag{10}$$

and the invariant $\delta^j(v) = \delta^j_X(v)$ defined by

$$\begin{aligned} \delta^j_X(v) = \sum_i h^j(\mathcal{I}_X(n_{1,i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{2,i}+v)) \\ + \sum_i h^j(\mathcal{I}_X(n_{3,i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{4,i}+v)). \end{aligned} \tag{11}$$

Proposition 4.1. *Let X be any surface in $\mathbb{P}^4$ of degree d and sectional genus π. Then the following expressions are equal*

$$\begin{aligned} {}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I) + {}_0\mathrm{ext}^3_R(I,I) = 1 - \delta^0(0) = \chi(\mathcal{N}_X) - \delta^0(-5) \\ = \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5). \end{aligned} \tag{12}$$

Moreover

$$\chi(\mathcal{N}_X(v)) = dv^2 + 5dv + 5(2d + \pi - 1) - d^2 + 2\chi(\mathcal{O}_X). \tag{13}$$

Proof. The first upper equality follows easily by applying ${}_v\mathrm{Hom}_R(-, I)$ (for $v = 0$) to the resolution (10) because $\mathrm{Hom}_R(I, I) \simeq R$ and because the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups. Similarly we compute $\delta^0(-5)$ which through the duality (3) leads to the alternating sum of ${}_0\mathrm{ext}^i_{\mathfrak{m}}(I, I)$. Combining with (5), recalling $\mathcal{H}om_{\mathcal{O}}(\mathcal{I}_X, \mathcal{I}_X) \cong \mathcal{O}$ and $\mathcal{E}xt^1_{\mathcal{O}}(\mathcal{I}_X, \mathcal{I}_X) \cong \mathcal{N}_X$, we get the next equality in the first line. The other equalities involving $\delta^j(v)$ follow from (2), (3) and (4) as outlined in [33], Lem 2.2 in the curve case. The surface case is technically more complicated because the spectral sequence of the proof, $E_2^{p,q} = {}_v\mathrm{Ext}^p_R(I, H^q_{\mathfrak{m}}(I))$, contains one more non-vanishing term. The principal parts of the proof are, however, the same, and we leave this part to the reader. Similarly the arguments of [33], Rem 2.4, lead to the formula

$$\chi(\mathcal{N}_X(v)) = \chi(\mathcal{O}_X(v)) + \chi(\mathcal{O}_X(-v-5)) - d^2 \tag{14}$$

for any surface X, from which (13) of Proposition 4.1 easily follows provided we combine with (1). Since we do not have a reference of (13) in the generality of an *arbitrary surface* (i.e., locally Cohen-Macaulay and equidimensional, see Remark below) and since the arguments of [33], Rem 2.4 was only sketched, we will include a proof of (14).

Firstly, we compute $\chi(\mathcal{O}_X(v)) = \chi(\mathcal{O}\ \ (v)) - \chi(\mathcal{I}_X(v))$, $\chi(\mathcal{O}\ \ (v)) = \binom{v+4}{4}$, directly from (10) as a large sum of binomials. Recalling that $\chi(\mathcal{O}_X(v))$ is the polynomial (1) of degree 2, we get

$$\sum_{j=1}^{4}(-1)^{j-1}r_j = 1\ ,\ \sum_{j=1}^{4}(-1)^{j-1}\sum_i n_{j,i} = 0\ \text{ and }\ \sum_{j=1}^{4}(-1)^{j-1}\sum_i n_{j,i}^2 = -2d\ . \tag{15}$$

Now as in the very first part of the proof, we apply ${}_v\mathrm{Hom}_R(-,I)$ to (10). Since we get ${}_v\mathrm{Ext}^i_R(I,I) \cong\ H^{i-1}(\mathcal{N}_X(v))$ for $v \gg 0$ and $i \geq 1$ directly from (2), (3) and (4) and we have $\mathrm{Hom}_R(I,I) \simeq R$, we find

$$\dim R_v\ -\ \chi(\mathcal{N}_X(v))\ =\ \delta^0(v)\ =\ \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{I}_X(n_{j,i}+v))\ ,\ v \gg 0\ . \tag{16}$$

By (10),

$$\begin{aligned}\chi(\mathcal{I}_X(-v-5)) &= \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}\ \ (-n_{j,i}-v-5))\\ &= \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}\ \ (n_{j,i}+v)).\end{aligned}$$

The right-hand side of (16) is therefore equal to

$$\chi(\mathcal{I}_X(-v-5)) - \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v)).$$

Then we compute $\sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v))$ by just using (1) and (15). We get exactly

$$\sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v)) = \chi(\mathcal{O}_X(v)) - d^2,$$

and (16) translates to $\dim R_v - \chi(\mathcal{N}_X(v)) = \chi(\mathcal{I}_X(-v-5)) - \chi(\mathcal{O}_X(v)) + d^2$ and we get (14). □

Remark 4.2. Note that the formula (13) of Proposition 4.1 is certainly straightforward to prove for *smooth* surfaces by combining the well-known formula

$$\chi(\mathcal{N}_X(v)) = dv^2 + 5dv + 5(d-\pi+1) - 2K^2 + 14\chi(\mathcal{O}_X)$$

with the double point formula $d^2 - 10d - 5H.K - 2K^2 + 12\chi(\mathcal{O}_X) = 0$.

Now we come to the analogue of Theorem 3.4. Also in this case ${}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I)$ is a lower bound of $\mathrm{H}(d,p,\pi)_\gamma$. Since the basic part of the proof of the Theorem below is similar to the proof of Theorem 3.4, we will only sketch the proof. Note that in the surface case, we do not succeed so nicely as in the curve case because the lower bound above is not directly given by the first equality

of Proposition 4.1, due to the term ${}_0\mathrm{ext}^3_R(I,I)$. Since we have ${}_0\mathrm{Ext}^3_R(I,I) \cong {}_{-5}\mathrm{Ext}^2_{\mathfrak{m}}(I,I)^\vee \cong {}_{-5}\mathrm{Hom}_R(I,M_1)^\vee$ by (2) and (3) and $M_1 \cong H^2_{\mathfrak{m}}(I)$ we get at least

Proposition 4.3. *Let X be a surface in $\mathbb{P}^4$, let $M_i = H^i_*(\mathcal{I}_X)$ for $i = 1, 2$ and put $I = I_X$ and $\delta^j(v) = \delta^j_X(v)$ for any j and v. Then the dimension of $\mathrm{H}(d,p,\pi)$ at (X) satisfies*

$$\dim_{(X)} \mathrm{H}(d,p,\pi) \geq 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - \sum_i h^1(\mathcal{I}_X(n_{1,i} - 5)).$$

Moreover let X be a generic surface of a generically smooth component V of $\mathrm{H}(d,p,\pi)$ and suppose ${}_{-5}\mathrm{Hom}_R(I,M_2) = 0$. Then

$$\dim V = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - \sum_{i=0}^{1} {}_{-5}\mathrm{ext}^i_R(I,M_1).$$

Proof. For the inequality, we remark that

$${}_0\mathrm{ext}^3_R(I,I) = {}_{-5}\mathrm{hom}_R(I,M_1) \leq \sum_i h^1(\mathcal{I}_X(n_{1,i} - 5))$$

because ${}_{-5}\mathrm{Hom}_R(I,M_1)$ is the kernel of the map $\oplus_i H^1(\mathcal{I}_X(n_{1,i} - 5)) \longrightarrow \oplus_i H^1(\mathcal{I}_X(n_{2,i} - 5))$ induced by the corresponding map in (10). We conclude by Proposition 4.1.

To find $\dim V$ we proceed as in the proof of Theorem 3.4 (see the last part of the proof of Theorem 3.7 for a close idea), and we get $\dim V = {}_0\mathrm{ext}^1_R(I,I)$, i.e.,

$$\dim V = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) + {}_0\mathrm{ext}^2_R(I,I) - {}_0\mathrm{ext}^3_R(I,I).$$

By (3) we have ${}_0\mathrm{ext}^2_R(I,I) = {}_{-5}\mathrm{ext}^3_{\mathfrak{m}}(I,I)$ and we conclude by the exact sequence associated to (2),

$$\begin{aligned} 0 \to {}_{-5}\mathrm{Ext}^1_R(I,H^2_{\mathfrak{m}}(I)) \to {}_{-5}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to {}_{-5}\mathrm{Hom}_R(I,H^3_{\mathfrak{m}}(I)) \\ \to {}_{-5}\mathrm{Ext}^2_R(I,H^2_{\mathfrak{m}}(I)) \to \quad . \end{aligned} \tag{17}$$

□

Under more specific assumptions we are able to prove,

Proposition 4.4. *Let X be any surface in $\mathbb{P}^4$ and suppose*

$${}_0\mathrm{Hom}_R(I,M_1) = {}_{-5}\mathrm{Ext}^1_R(I,M_1) = {}_{-5}\mathrm{Hom}_R(I,M_2) = 0.$$

Then X is unobstructed and

$$\dim_{(X)} \mathrm{H}(d,p,\pi) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - {}_{-5}\mathrm{hom}_R(I,M_1).$$

Proof. Due to [27], Rem.3.7 (cf. [49], Thm.2.1), $\mathrm{H}(d,p,\pi)_\gamma \cong \mathrm{H}(d,p,\pi)$ at (X) provided ${}_0\mathrm{Hom}_R(I,M_1) = 0$. Then we see by the arguments of (17) that ${}_0\mathrm{Ext}^2_R(I,I) = 0$. It follows that $\mathrm{H}(d,p,\pi)_\gamma$ is smooth at (X) of dimension ${}_0\mathrm{ext}^1_R(I,I)$. Then we conclude by Proposition 4.1. □

Remark 4.5.

(i) Proposition 4.4 is mainly proved in [30], Sect. 1. In [30] we moreover use (2) and (3) to prove a vanishing result for $H^1(\mathcal{N}_X)$. Indeed we show that $H^1(\mathcal{N}_X) = 0$ provided

$$H^1(\mathcal{I}_X(n_{2,i})) = H^1(\mathcal{I}_X(n_{2,i} - 5)) = 0 \quad \text{and}$$
$$H^2(\mathcal{I}_X(n_{1,i})) = H^2(\mathcal{I}_X(n_{1,i} - 5)) = 0$$

for every i.

(ii) Let X be an arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$. Then $M_1 = M_2 = 0$ and $\delta^1(v) = \delta^2(v) = 0$ for every v and we can use Proposition 4.4 to see that X is unobstructed and $\dim_{(X)} \mathrm{H}(d, p, \pi) = 1 + \delta^3(-5) = 1 - \delta^0(0)$. This coincides with [13].

We will illustrate the results of this section by an example. If the assumptions of Proposition 4.4 or Remark 4.5 are not satisfied, then the surface may be obstructed, and we refer to Section 8 for such examples.

Example 4.6. Let X be the smooth rational surface with invariants $d = 11$, $\pi = 11$ (no 6-secant) and $K^2 = -11$ (cf. [43] or [11], B1.17, see also [16]). In this case the graded modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ are supported at two consecutive degrees and satisfy

$$\dim H^1(\mathcal{I}_X(3)) = 2, \qquad \dim H^2(\mathcal{I}_X(1)) = 3,$$
$$\dim H^1(\mathcal{I}_X(4)) = 1, \qquad \dim H^2(\mathcal{I}_X(2)) = 1.$$

Moreover $I = I_X$ admits a minimal resolution (cf. [11])

$$0 \to R(-9) \to R(-8)^{\oplus 3} \oplus R(-7)^{\oplus 3} \to R(-7)^{\oplus 2} \oplus R(-6)^{\oplus 12} \to R(-5)^{\oplus 10} \to I \to 0.$$

It follows that ${}_{-5}\mathrm{Hom}_R(I, M_2) = 0$ and ${}_{-5}\mathrm{Ext}^i_R(I, M_1) = 0$ for $i = 0, 1$. By Proposition 4.4, $\mathrm{H}(d, p, \pi)$ is smooth at (X) and

$$\begin{aligned}\dim_{(X)} \mathrm{H}(d, p, \pi) &= 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5) \\ &= 1 + 12h^2(\mathcal{I}_X(1)) - h^2(\mathcal{I}_X(2)) + 3h^1(\mathcal{I}_X(3)) - h^1(\mathcal{I}_X(4)) = 41.\end{aligned}$$

In this example it is, however, easier to use Proposition 4.1 to get

$$\begin{aligned}1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) &= \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) \\ &= 5(2d + \pi - 1) - d^2 + 2\chi(\mathcal{O}_X) = 41\end{aligned}$$

because $\delta^i(0)$ for $i > 0$ is easily seen to be zero. We may also use Remark 4.5 to see $H^1(\mathcal{N}_X) = 0$. Since any smooth surface satisfies

$$H^2(\mathcal{N}_X) = 0 \quad \text{provided} \quad H^2(\mathcal{O}_X(1)) = 0$$

(due to the existence of the natural surjection $\mathcal{O}_X(1)^5 \to \mathcal{N}_X$), we may conclude as above directly from $\dim H^0(\mathcal{N}_X) = \chi(\mathcal{N}_X) = 41$.

One may hope that a generalization of Theorem 3.7 to surfaces will contain a more complete result. To do it we need to generalize some of the theorems in [38] to surfaces. This will be done in the next two sections. The biliaison statements of Remark 3.9 will be generalized to any codimension 2 lCM equidimensional subscheme of $\mathbb{P}^{n+2}$ and carried out in later sections.

5. The smoothness of the "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_{\rho}$

In this section we prove the local smoothness of the "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_{\rho} :=$ isomorphism classes of graded R-modules M_1 and M_2 satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, given by sending the surface X onto the class of the triple (M_1, M_2, b) where $M_i = H^i_*(\mathcal{I}_X)$ and $b \in {}_0\mathrm{Ext}^2_R(M_2, M_1)$ is the extension determined by X (cf. Remark 5.2 (ii)). To prove our theorem we first take in Proposition 5.1 a close look to Bolondi's short exact "resolution" of the homogeneous ideal of a surface X ([6]) and how we can define the extension b given in Horrock's paper [24]. As in [11] the ideal is the cokernel of some syzygy modules of M_1 and M_2, up to direct free factors. The proposition somehow uses and extends a result of Rao for a curve C, namely that the minimal resolution of I_C can be put in the following form

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \to F_1 \to I_C \to 0 \tag{18}$$

where $0 \to L_4 \xrightarrow{\sigma} L_3 \to \cdots \to M \to 0$ is a minimal resolution of M and F_i are free modules ([47]). Moreover we use local flatness criteria to generalize Bolondi's construction in [6] so that it works for flat resolutions over a local ring, rather than over a field. This is also the approach of [23] in the curve case.

Let X be a surface in $\mathbb{P}^4$ and let

$$\begin{aligned} &0 \to P_5 \xrightarrow{\sigma_5} P_4 \xrightarrow{\sigma_4} P_3 \xrightarrow{\sigma_3} \cdots \longrightarrow P_0 \xrightarrow{\sigma_0} M_1 \to 0, \\ &0 \to Q_5 \xrightarrow{\tau_5} Q_4 \xrightarrow{\tau_4} Q_3 \xrightarrow{\tau_3} \cdots \longrightarrow Q_0 \xrightarrow{\tau_0} M_2 \to 0 \end{aligned} \tag{19}$$

(for short $\sigma_\bullet : P_\bullet \to M_1 \to 0$ and $\tau_\bullet : Q_\bullet \to M_2 \to 0$) be minimal free resolutions over R. Let $K_\bullet$ and $L_\bullet$ be the syzygies of M_1 and M_2 respectively, i.e., $K_i = \ker \sigma_i$ and $L_i = \ker \tau_i$. Recall that syzygies have nice cohomological properties ([11], [6]), for instance

$$\begin{aligned} &M_1 = H^1_*(\tilde{K}_1) \quad \text{and} \quad H^2_*(\tilde{K}_1) = H^3_*(\tilde{K}_1) = 0, \\ &M_2 = H^3_*(\tilde{L}_3) \quad \text{and} \quad H^1_*(\tilde{L}_3) = H^2_*(\tilde{L}_3) = 0. \end{aligned} \tag{20}$$

There is a strong connection between the resolutions (19), the minimal resolution (10) of $I = I_X$ and the following minimal resolutions of $A = H^0_*(\mathcal{O}_X)$;

$$0 \to P'_3 \xrightarrow{\sigma'_3} P'_2 \xrightarrow{\sigma'_2} P'_1 \xrightarrow{\sigma'_1} P_0 \oplus R \to A \to 0 \tag{21}$$

where the morphism $P_0 \oplus R \to A$ of (21) is naturally deduced from $P_0 \to M_1$ of (19) and the exact sequence $R \to A \to M_1 \to 0$ and where $\sigma'_\bullet : P'_\bullet \to \ker(P_0 \oplus R \to A) \to 0$ is a minimal R-free resolution (cf. [38], p. 46). The connection we have in mind can be formulated and proved for a *family* of surfaces with constant

cohomology, at least locally, e.g., we can replace the field k by a local k-algebra S with residue field k. Now, in [6], Bolondi uses some ideas of Horrocks [24] to define an element $b \in {}_0\mathrm{Ext}^2_R(M_2, M_1)$ and the "Horrocks triple" $D =: (M_1, M_2, b)$ associated to X such that, conversely given $D = (M_1, M_2, b)$ where M_i are R-modules of finite length, there is a surface X whose homogeneous ideal I is defined in the following way. For some integer $h \in \mathbb{Z}$ there is an exact sequence $0 \to L'_3 \to K'_1 \to I(h) \to 0$ where L'_3 (resp. K'_1) is isomorphic to the syzygy L_3 (resp. K_1) up to some R-free module F_L (resp. F_K). Up to biliaison this construction is the inverse to the first approach which defines (M_1, M_2, b) from a given X. To prove the main smoothness theorem of this section, we need to adapt the approach above by determining F_L and F_K more explicitly and such that it works over (at least an artinian) S. Using also ideas of Rao's paper [47], we can prove

Proposition 5.1. *Let X be a surface in $\mathbb{P}^4_S$, flat over a local noetherian k-algebra S with residue field k, and suppose that $M_1 = H^1_*(\mathcal{I}_X)$, $M_2 = H^2_*(\mathcal{I}_X)$ and $I = I_X$ are flat S-modules. Then there exist minimal R-free resolutions of M_i, I and $A = H^0_*(\mathcal{O}_X)$ as in* (19), (10) *and* (21), *with $R = S[X_0, X_1, \ldots, X_4]$. Moreover let $L'_3 = \ker \sigma'_1$ and let K'_1 be the kernel of the composition of σ'_1 and the natural projection $P_0 \oplus R \to P_0$, cf.* (21). *Then there is an exact sequence*

$$0 \longrightarrow L'_3 \xrightarrow{b'} K'_1 \longrightarrow I \longrightarrow 0 \tag{22}$$

of flat graded S-modules and a surjective morphism $d : {}_0\mathrm{Hom}_R(L'_3, K'_1) \longrightarrow {}_0\mathrm{Ext}^2_R(M_2, M_1)$, defining a triple (M_1, M_2, b) where $b = d(b')$, coinciding with the uniquely defined "Horrocks triple" of [24] *or* [6]. *Moreover L'_3 (resp. K'_1) is the direct sum of a 3rd syzygy of M_2 (resp. 1st syzygy of M_1) up to a direct free factor, i.e., there exist R-free modules F_L and F_K such that the horizontal exact sequences in the diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & K'_1 & \longrightarrow & P'_1 & \longrightarrow & P_0 \\ & \downarrow & \circ & \downarrow & \circ & \| \\ 0 \to & K_1 \oplus F_K & \to & P_1 \oplus F_K & \xrightarrow{\sigma_1 \oplus 0} & P_0 \end{array}$$

are isomorphic (i.e., the downarrows are isomorphisms). Similarly, the exact sequences $0 \to Q_5 \xrightarrow{(\tau_5, 0)} Q_4 \oplus F_L \to L_3 \oplus F_L \to 0$ and $0 \to P'_3 \to P'_2 \to L'_3 \to 0$ are isomorphic as well.

Remark 5.2.

(i) By a surface $X \subseteq \mathbb{P}^4_S$ in Proposition 5.1 we actually mean that $X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(k)$ is a surface (i.e., locally Cohen-Macaulay and equidimensional of dimension 2).

(ii) The proposition above, defining the "Horrocks triple" (M_1, M_2, b) from a given $X \subseteq \mathbb{P}^4_S$, can be regarded as our definition of the "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho =$ isomorphism classes of graded R-modules M_1 and M_2 satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b.

Proof. We obviously have minimal resolutions of $M_i \otimes_S k$, $I_X \otimes_S k$ and $A \otimes_S k$ as described above with $R = k[X_0, X_1, \ldots, X_4]$, cf. (19), (10) and (21). These resolutions can easily be lifted to the minimal resolution of the proposition by cutting into short exact sequences and using the flatness of the modules involved.

By the definition of L_3' and K_1' there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & R & \longrightarrow & R \longrightarrow 0 \\ & & & & & & \downarrow & \circ & \downarrow \\ 0 \longrightarrow & L_3' & \longrightarrow & P_1' & \longrightarrow & P_0 \oplus R & \longrightarrow & A & \longrightarrow 0 \\ & & & \| & \circ & \downarrow & \circ & \downarrow & \\ 0 \longrightarrow & K_1' & \longrightarrow & P_1' & \longrightarrow & P_0 & \longrightarrow & M_1 & \longrightarrow 0 \end{array}$$

and we get the exact sequence (22) by the snake lemma. Comparing the lower exact sequence in the last diagram with the following part of the minimal resolution of M_1; $\to P_1 \to P_0 \to M_1 \to 0$, we get the commutative diagram of the proposition because K_1 is the 1st syzygy of M_1.

To prove the corresponding commutative diagram for L_3' and L_3, we sheafify (22), and we get $M_2 \simeq H^3_*(\tilde{L}_3')$. Recalling the definition of L_3', we get the exact sequence

$$H^4_*(\tilde{P}_2')^\vee \to H^4_*(\tilde{P}_3')^\vee \to M_2^\vee \simeq \mathrm{Ext}^5_R(M_2, R(-5)) \to 0$$

which we compare to the *minimal* resolution

$$Q_4^\vee \to Q_5^\vee \to \mathrm{Ext}^5_R(M_2, R) \to 0$$

obtained by applying $\mathrm{Hom}_R(-, R)$ to the resolution $Q_\bullet \to M_2$.

Recalling $H^4_*(\tilde{P}_i')^\vee(5) \simeq P_i'^\vee$, we get the conclusion, as in the proof of Thm. 2.5 of [47].

Finally to define the morphism d and to see that the defined triple (M_1, M_2, b) corresponds the one given by Horrocks' construction (seen to be unique by [24]), one may consult [6] for the case $S = k$ which, however, generalizes to a local ring S. The important part is as follows. The definition of K_1' and K_0 implies

$$\mathrm{Ext}^2(M_2, M_1) \simeq \mathrm{Ext}^3(M_2, K_0) \simeq \mathrm{Ext}^4(M_2, K_1').$$

Next, by Gorenstein duality, we know $\mathrm{Ext}^i_R(M_2, R) = 0$ for $i \neq 5$. Hence the definition of the syzygies L_i leads to $\mathrm{Ext}^4(M_2, K_1') \simeq \mathrm{Ext}^3(L_0, K_1') \simeq \mathrm{Ext}^1(L_2, K_1')$ and to a diagram

$$\begin{array}{ccccc} {}_0\mathrm{Hom}_R(Q_3, K_1') & \to & {}_0\mathrm{Hom}(L_3, K_1') & \to & {}_0\mathrm{Ext}^1(L_2, K_1') \to 0 \\ & & \downarrow & & \downarrow \\ & & {}_0\mathrm{Hom}(L_3', K_1') & & {}_0\mathrm{Ext}^2_R(M_2, M_1) \end{array} \tag{23}$$

where the horizontal sequence is exact and the first (resp. second) vertical map is injective and split (resp. an isomorphism). We let $d : {}_0\mathrm{Hom}_R(L_3', K_1') \to {}_0\mathrm{Ext}^2_R(M_2, M_1)$ be the obvious composition, first using the "inverse" of the split map, and we get the conclusions of the proposition. □

Now we will show the smoothness of φ. Indeed using Proposition 5.1 for S artinian, we get a rather easy proof of

Theorem 5.3. *The "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho$ = isomorphism classes of graded R-modules M_1 and M_2 satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, is smooth (i.e., for any surface X in $\mathbb{P}^4_k$, the corresponding local deformation functor of φ, given by $(X_S \subseteq \mathbb{P}^4_S) \mapsto$ class of (M_{1S}, M_{2S}, b_S), see right below, is formally smooth).*

Proof. Let $T \to S \to k$ be surjections of local artinian k-algebras with residue fields k such that $\ker(T \to S)$ is a k-module via $T \to k$. Let $X_S \subseteq \mathbb{P}^4_S$ be a deformation of $X \subseteq \mathbb{P}^4$ to S with constant postulation γ and constant deficiency $\rho = (\rho_1, \rho_2)$. Let (M_{1S}, M_{2S}, b_S) be the "Horrocks triple" defined by X_S (cf. Proposition 5.1). Note that M_{iS} for $i = 1, 2$ are S-flat by the definition of $H_{\gamma,\rho}$. Let (M_{1T}, M_{2T}, b_T) be a given deformation of (M_{1S}, M_{2S}, b_S) to T. To prove the smoothness at (X), we must show the existence of a deformation $X_T \subseteq \mathbb{P}^4_T$ of $X_S \subseteq \mathbb{P}^4_S$, whose corresponding "Horrocks triple" is precisely (M_{1T}, M_{2T}, b_T), modulo graded isomorphisms of (M_{1T}, M_{2T}) commuting with b_T.

We have by Proposition 5.1 minimal resolutions of M_{iS}, I_{X_S} and A_S over $R_S := S[X_0, X_1, \ldots, X_4]$ as in (10), (19)–(21) and flat S-modules L_{iS}, K_{iS}, L'_{3S}, K'_{1S} fitting into the exact sequence (22) and a surjection d defined as the composition (cf. (23))

$$\begin{array}{ccccccc} {}_0\mathrm{Hom}_{R_S}(L'_{3S}, K'_{1S}) & \to & {}_0\mathrm{Hom}_{R_S}(L_{3S}, K'_{1S}) & \to & {}_0\mathrm{Ext}^1_{R_S}(L_{2S}, K'_{1S}) \simeq {}_0\mathrm{Ext}^2_{R_S}(M_{2S}, M_{1S}) & & \\ \cup & & \cup & & \cup & & \cup \\ b'_S & \longrightarrow & \beta_S & \longrightarrow & b_S & \longrightarrow & b_S \end{array} \tag{24}$$

"on the S-level" (β_S is simply the image of b'_S via the map of (24)) which lifts the corresponding resolutions/modules/sequences on the "k-level". Since M_{iT} are given deformations of M_{iS}, we can lift the minimal resolutions $\sigma_{\bullet S} : P_{\bullet S} \to M_{1S}$ and $\tau_{\bullet S} : Q_{\bullet S} \to M_{2S}$ further to T, thus proving the existence of deformations L_{iT}, K_{iT}, L'_{3T}, K'_{1T} of L_{iS}, K_{iS}, L'_{3S}, K'_{1S} resp. (the free submodules F_{LS} and F_{KS} of L'_{3S} and K'_{1S} are lifted trivially). So we have a diagram (23) and hence a sequence (24) "on the T-level" where the elements b'_T and β_T are not yet defined. The element $b_T \in {}_0\mathrm{Ext}^1(L_{2T}, K'_{1T}) \simeq {}_0\mathrm{Ext}^2_{R_T}(M_{2T}, M_{1T})$ is, however, given and if we consider the diagram (cf. (23))

$$\begin{array}{ccccccc} {}_0\mathrm{Hom}_{R_T}(Q_{3T}, K'_{1T}) & \to & {}_0\mathrm{Hom}_{R_T}(L_{3T}, K'_{1T}) & \to & {}_0\mathrm{Ext}^1_{R_T}(L_{2T}, K'_{1T}) & \to & 0 \\ \downarrow & & \circ & & \downarrow \alpha & \circ & \downarrow \\ {}_0\mathrm{Hom}_{R_S}(Q_{3S}, K'_{1S}) & \to & {}_0\mathrm{Hom}_{R_S}(L_{3S}, K'_{1S}) & \to & {}_0\mathrm{Ext}^1_{R_S}(L_{2S}, K'_{1S}) & \to & 0 \end{array}$$

of exact horizontal sequences and surjective vertical maps deduced from $0 \to L_{3T} \to Q_{3T} \to L_{2T} \to 0$, we easily get a morphism $\beta_T \in {}_0\mathrm{Hom}(L_{3T}, K'_{1T})$ such that $\alpha(\beta_T) = \beta_S$, i.e., $\beta_T \otimes_T S = \beta_S$. Since $L'_{3S} \simeq L_{3S} \oplus F_{LS}$ we can decompose the map b'_S as $(\beta_S, \gamma_S) \in {}_0\mathrm{Hom}(L'_{3S}, K'_{1S})$, and taking any lifting $\gamma_T : F_{LT} \to K'_{1T}$ of γ_S, we get a map $b'_T = (\beta_T, \gamma_T) \in {}_0\mathrm{Hom}(L'_{3T}, K'_{1T})$ fitting into a commutative

diagram

$$\begin{array}{ccc} L_{3T} \oplus F_{LT} \simeq L'_{3T} & \xrightarrow{b'_T} & K'_{1T} \\ \downarrow & \circ & \downarrow \\ L_{3S} \oplus F_{LS} \simeq L'_{3S} & \xrightarrow{b'_S} & K'_{1S} \ . \end{array}$$

Once having proved the existence of such a commutative diagram, we can define a surface X_T of $\mathbb{P}^4_T$ with the desired properties, thus proving the claimed smoothness. Indeed it is straightforward to see that $\operatorname{coker} b'_T$ is a (flat) deformation of $\operatorname{coker} b'_S = I_{X_S}$ to T. Moreover one knows that an $R_T := T[X_0, X_1, \ldots, X_4]$-*module* $\operatorname{coker} b'_T$ which lifts a graded ideal I_{X_S} is again a *graded ideal* I_T (we can deduce this information by interpreting the isomorphisms $H^{i-1}(\mathcal{N}_X) \simeq \operatorname{Ext}^i_{\mathcal{O}}\ (\mathcal{I}_X, \mathcal{I}_X)$ for $i = 1, 2$ in terms of their deformation theories from which we see that $\widetilde{\operatorname{coker}}\, b'_T$ is a sheaf ideal, and we conclude by taking global sections, cf. [49] or [33], Lem. 4.8 for further details). Hence we have proved the existence of a surface $X_T = \operatorname{Proj}(R_T/I_T)$, flat over T which via $T \to S$ reduces to X_S. By the construction above the corresponding "Horrocks triple" is precisely the given triple (M_{1T}, M_{2T}, b_T), and we are done. □

Corollary 5.4. *Let X be a surface in $\mathbb{P}^4$. If the local deformation functors $Def(M_i)$ of M_i are formally smooth (for instance if ${}_0\operatorname{Ext}^2_R(M_i, M_i) = 0$) for $i = 1, 2$, and if*

$${}_0\operatorname{Ext}^3_R(M_2, M_1) = 0,$$

then $\mathrm{H}_{\gamma,\rho}$ is smooth at (X).

Proof. With notations as in the very first part of the proof of Theorem 5.3, it suffices to prove that there always exists a deformation (M_{1T}, M_{2T}, b_T) of (M_{1S}, M_{2S}, b_S) since then the proof above shows the existence of a deformation $X_T = \operatorname{Proj}(R_T/I_T)$ which reduces to X_S via $T \to S$. Since $Def(M_i)$ are formally smooth, it suffices to show the existence of b_T which maps to $b_S \in {}_0\operatorname{Ext}^2_{R_S}(M_{2S}, M_{1S})$. Let $\mathfrak{a} = \ker(T \to S)$. If we apply ${}_0\operatorname{Hom}_{R_T}(M_{2T}, -)$ to the exact sequence

$$0 \to \mathfrak{a} \otimes_T M_{1T} \cong \mathfrak{a} \otimes_k M_1 \to M_{1T} \to M_{1S} \to 0$$

and use ${}_0\operatorname{Ext}^3_R(M_2, M_1) = 0$, we see that ${}_0\operatorname{Ext}^2_{R_T}(M_{2T}, M_{1T}) \to {}_0\operatorname{Ext}^2_{R_T}(M_{2T}, M_{1S})$ is surjective. Hence we get a surjective map

$${}_0\operatorname{Ext}^1_{R_T}(L_{3T}, K'_{1T}) \simeq {}_0\operatorname{Ext}^2_{R_T}(M_{2T}, M_{1T}) \to {}_0\operatorname{Ext}^1_{R_S}(L_{2S}, K'_{1S}) \simeq {}_0\operatorname{Ext}^2_{R_S}(M_{2S}, M_{1S})$$

and we are done. □

Remark 5.5. If we, as in [38] for curves, had proven the existence of the "fiber" $\mathrm{H}_{\gamma,D}$, $D = (M_1, M_2, b)$, of φ as a scheme, then Theorem 5.3 must imply the smoothness of $\mathrm{H}_{\gamma,D}$ while [8] implies its irreducibility. Indeed [8], Cor. 3.2 tells that the family of surfaces in $\mathbb{P}^4$ belonging to the same shift of the same liaison class, with fixed postulation, form an irreducible family, from which we see that $\mathrm{H}_{\gamma,D}$ is irreducible. Note that we can work with $\mathrm{H}_{\gamma,D}$ as a locally closed subset of

$\mathrm{H}_{\gamma,\rho}$ (cf. the arguments of [4], Cor. 2.2, and combine with Proposition 5.1), even though we have not proved that φ extends to a morphism of representable functors.

6. The tangent space of $\mathrm{H}_{\gamma,\rho}$

In this section we determine the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X) and we give a criterion for $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d,p,\pi)$ to be isomorphic as schemes at (X). We end this section by considering an example.

Let X be a surface in $\mathbb{P}^4$ with graded ideal $I = I_X$ and let $D = (M_1, M_2, b)$, $M_i = H^i_*(\tilde{I})$, be its "Horrocks triple". Recall that ${}_0\mathrm{Ext}^1_R(I,I)$ is the *tangent space* of H_γ at (X) because a deformation in H_γ keeps the postulation constant, i.e., it corresponds precisely to a graded deformation of I [38]. Moreover there exist maps

$$\varphi_i : {}_0\mathrm{Ext}^1_R(I,I) \to {}_0\mathrm{Hom}_R(H^i_*(\tilde{I}), H^{i+1}_*(\tilde{I}))$$

taking an extension $0 \to I \to E \to I \to 0$ of ${}_0\mathrm{Ext}^1_R(I,I)$ onto the connecting homomorphism δ^i in the exact sequence

$$H^i_*(\tilde{E}) \to H^i_*(\tilde{I}) \xrightarrow{\delta^i} H^{i+1}_*(\tilde{I}) \to H^{i+1}_*(\tilde{E}).$$

For saturated homogeneous ideals we have $I = H^0_*(\tilde{I})$, and it follows that the composition $E \to H^0_*(\tilde{E}) \to H^0_*(\tilde{I})$ is surjective, i.e., we get $\varphi_0 = 0$. Moreover note that if δ^{i-1} and δ^i are both zero for some i, then the exact sequence $0 \to I \to E \to I \to 0$ above defines an extension

$$0 \to H^i_*(\tilde{I}) \to H^i_*(\tilde{E}) \to H^i_*(\tilde{I}) \to 0.$$

Since $M_i = H^i_*(\tilde{I})$ for $i = 1, 2$ and $E = H^3_*(\tilde{I})$, there are well-defined morphisms

$$\psi_i : \ker(\varphi_1, \varphi_2) \to {}_0\mathrm{Ext}^1_R(M_i, M_i) \quad \text{for} \quad \mathrm{i} = 1, 2$$

where $(\varphi_1, \varphi_2) : {}_0\mathrm{Ext}^1_R(I,I) \to {}_0\mathrm{Hom}(M_1, M_2) \times {}_0\mathrm{Hom}(M_2, E)$ and φ_i are defined above. Recalling $\rho = (\rho_1, \rho_2)$ we put

$$ {}_0\mathrm{Ext}^1_R(I,I)_\rho := \ker(\varphi_1, \varphi_2). \tag{25}$$

Using base change theorems, as in [38], we easily show that $\ker(\varphi_1, \varphi_2)$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X), i.e., we get

Proposition 6.1. *${}_0\mathrm{Ext}^1_R(I,I)_\rho$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X). In particular if*

$$ {}_0\mathrm{Hom}_R(I, M_1) = 0, \quad {}_0\mathrm{Hom}_R(M_1, M_2) = 0 \quad \textit{and} \quad {}_0\mathrm{Hom}(M_2, E) = 0, \tag{26}$$

then the tangent spaces of $\mathrm{H}_{\gamma,\rho}, \mathrm{H}_\gamma$ and $\mathrm{H}(d,p,\pi)$ are isomorphic at (X). Indeed $\mathrm{H}_\gamma \cong \mathrm{H}(d,p,\pi)$ as schemes at (X), and if $\mathrm{H}_{\gamma,\rho}$ is smooth at (X), then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma$ are isomorphic as schemes at (X) as well.

Proof. As earlier remarked, cf. (2) and (5), ${}_0\mathrm{Ext}^1_R(I,I) \cong \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \cong H^0(\mathcal{N}_X)$ provided ${}_0\mathrm{Hom}_R(I, M_1) = 0$. Moreover ${}_0\mathrm{Ext}^1_R(I,I)_\rho \cong {}_0\mathrm{Ext}^1_R(I,I)$ since $\varphi_i = 0$ for $i = 1, 2$.

For the isomorphism as schemes we remark that $\mathrm{H}_\gamma \simeq \mathrm{H}(d,p,\pi)$ folloed from [27], Thm. 3.6 and Rem. 3.7 (see [49] and [33], proof of Thm 2.6 (i) for details). Finally if $\mathrm{H}_{\gamma,\rho}$ is smooth at (X), then the embedding $\mathrm{H}_{\gamma,\rho} \hookrightarrow \mathrm{H}_\gamma$ is smooth at (X) (since the tangent map is surjective), hence etale, hence an isomorphism at (X) since the embedding is universally injective. □

Remark 6.2. If we suppose (26), then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma$ are isomorphic as schemes at (X) by [31], Thm. 3.7 without requiring the smoothness of $\mathrm{H}_{\gamma,\rho}$ at (X). See also Remark 9.3.

In [31] we also gave almost complete proofs of Remark 6.2 and of the following two non-trivial results (cf. [31], Prop. 3.4 and Prop. 3.6). Note that Remark 6.3 generalizes Corollary 5.4.

Remark 6.3. Let X be a surface in $\mathbb{P}^4$. Then for $i = 1, 2$ there exist morphisms $e_i : {}_0\mathrm{Ext}^1_R(M_i, M_i) \to {}_0\mathrm{Ext}^3_R(M_2, M_1)$ and an induced morphism

$$\bar{e}_1 : {}_0\mathrm{Ext}^1_R(M_1, M_1) \to {}_0\mathrm{Ext}^3_R(M_2, M_1)/e_2({}_0\mathrm{Ext}^1_R(M_2, M_2))$$

such that if the local deformation functors $Def(M_i)$ of M_i are formally smooth (for instance if ${}_0\mathrm{Ext}^2_R(M_i, M_i) = 0$) for $i = 1, 2$, and if the morphism $\bar{e}_1$ is surjective, then V_ρ is smooth at $D = (M_1, M_2, b)$ (i.e., the local deformation functor of D is formally smooth).

Remark 6.4. Let X be a surface in $\mathbb{P}^4$ and let $\epsilon = \dim \operatorname{coker} \bar{e}_1$. Then

$$\dim {}_0\mathrm{Ext}^1_R(I, I)_\rho = 1 + \delta^3(-5) + \sum_{i=0}^{3}(-1)^i \, {}_0\mathrm{ext}^i_R(M_2, M_1) - \sum_{i=0}^{1}(-1)^i \, {}_0\mathrm{ext}^i_R(M_1, M_1) - \sum_{i=0}^{1}(-1)^i \, {}_0\mathrm{ext}^i_R(M_2, M_2) + \epsilon.$$

To illustrate the results we have proved, we consider an example of a surface X of $\mathbb{P}^4$ where actually V_ρ is smooth and non-trivial at the corresponding (M_1, M_2, b), cf. Corollary 5.4. Moreover all conditions of Proposition 6.1 are satisfied, and it follows that $\mathrm{H}_{\gamma,\rho}$ and $\mathrm{H}(d,p,\pi)$ are isomorphic and smooth at (X).

Example 6.5. Let X be the smooth elliptic surface with invariants $d = 11$, $\pi = 12$ and $K^2 = -4$ (cf. [43] or [11], B7.6). Then the graded modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ for $i = 1, 2$ vanish for every v except in the following cases

$$h^1(\mathcal{I}_X(3)) = 1, \quad h^2(\mathcal{I}_X(1)) = 2, \quad h^2(\mathcal{I}_X(2)) = 1.$$

Moreover $I = I_X$ admits a minimal resolution (cf. [11])

$$0 \to R(-8) \to R(-7)^{\oplus 6} \to R(-6)^{\oplus 13} \to R(-5)^{\oplus 8} \oplus R(-4) \to I \to 0.$$

It follows that ${}_0\mathrm{Ext}^i(M_j, M_j) = 0$ for $i \geq 2$ and $j = 1, 2$ and that ${}_0\mathrm{Ext}^3(M_2, M_1) = 0$. By Corollary 5.4 and Proposition 6.1 we get that $\mathrm{H}(d,p,\pi) \cong \mathrm{H}_{\gamma,\rho}$ are smooth at (X). If we, however, want to compute the dimension of $\mathrm{H}(d,p,\pi)$ at (X) and will avoid Remark 6.4 which we have not proved, we still have to use the results of Section 4. Let us only use the two "most general" results there, Proposition 4.1 and

Propositions 4.3, to illustrate the principle of semicontinuity a little extended (to include the semicontinuity of the graded Betti numbers). Let V be the generically smooth component of $\mathrm{H}(d,p,\pi)$ to which (X) belongs. Since $\mathrm{H}(d,p,\pi) \cong \mathrm{H}_{\gamma,\rho}$ at (X), then a generic surface $\tilde{X}$ of V also belongs to $\mathrm{H}_{\gamma,\rho}$. Inside H_γ, hence inside $\mathrm{H}_{\gamma,\rho}$, the graded Betti numbers of the homogeneous ideal of the surfaces obey semicontinuity by Remark 7(b) of [34]!! Since we from the minimal resolution of I_X can see that, for every i, $\beta_{j,i} \neq 0$ for at most one j and since the Hilbert functions of X and $\tilde{X}$ are the same, they have exactly the same graded Betti numbers. Moreover note that $h^i(\mathcal{I}_{\tilde{X}}(v)) = h^i(\mathcal{I}_X(v))$ for any i, v since X has seminatural cohomology. It follows that

$$\begin{aligned}\dim V &= 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5)\\ &= 1 + h^3(\mathcal{I}_X(-1)) + 8h^3(\mathcal{I}_X) + 13h^2(\mathcal{I}_X(1)) - 6h^2(\mathcal{I}_X(2)) - h^1(\mathcal{I}_X(3)) = 50.\end{aligned}$$

Since we have proved $\dim V = 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5)$ it is easier to use Proposition 4.1 to get

$$\dim V = \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) = 5(2d + \pi - 1) - d^2 + 2\chi(\mathcal{O}_X) = 50$$

because $\delta^i(0)$ for $i > 0$ is easily seen to be zero.

7. Linkage of surfaces

The main result of this section shows how to compute the dimension of $\mathrm{H}_{\gamma,\rho}$ and the dimension of its tangent space at (X) provided we know how to solve the corresponding problem for a linked surface X' (Theorem 7.1). In another related result (Proposition 7.4 with $c > 0$) we give conditions on, e.g., a generic surface of $\mathrm{H}(d,p,\pi)$ such that corresponding linked surface X' is non-generic in the sense $\dim_{(X')} \mathrm{H}_{\gamma',\rho'} < \dim_{(X')} \mathrm{H}(d',p',\pi')$. It follows that a new surface, the generic one with "smaller" cohomology, has to exist! Indeed recall that linkage is a well-known method for proving existence of surfaces with certain properties, e.g., see [42], [26], [40], [46], [12], [44], [1] to mention a few papers which use linkage in this way. In these and similar papers we see that the linked surface X' is usually generic if X is generic. In Remark 7.2 we notice that if certain cohomological assumptions, cf. (30), are satisfied, then X' is generic if and only if X is generic. Using Proposition 7.4 with $c > 0$, however, then some of the cohomology groups of (30) are non-zero, and under some assumptions we get the existence of a non-generic surface $X' \in \mathrm{H}_{\gamma',\rho'}$ and hence a generic one $\notin \mathrm{H}_{\gamma',\rho'}$ as well. In proving the results of this section we substantially need the theory of linkage of families developed in [29].

Since the main even liaison result of this paper, which we prove in the final section, requires that the linkage theorem of this section is proven for equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, we prove Theorem 7.1 in this generality. The other results and examples of this section deal, however, with surfaces.

Now, if the surfaces X and X' are (algebraically) linked by a complete intersection (a CI) Y of type (f,g), then the dualizing sheaf $\omega_{X'}$ satisfies $\omega_{X'} = \mathcal{I}_{X/Y}(f+g-5)$ where $\mathcal{I}_{X/Y} = \ker(\mathcal{O}_Y \to \mathcal{O}_X)$ ([45], [39]). Moreover $\omega_X = \mathcal{I}_{X'/Y}(f+g-5)$ and we get

$$\begin{aligned} \chi(\mathcal{O}_X(v)) + \chi(\mathcal{O}_{X'}(f+g-5-v)) &= \chi(\mathcal{O}_Y(v)) \\ h^i(\mathcal{I}_{X'}(v)) = h^{3-i}(\mathcal{I}_X(f+g-5-v)), &\quad \text{for } i=1 \text{ and } 2 \\ h^i(\mathcal{I}_{X'/Y}(v)) = h^{2-i}(\mathcal{O}_X(f+g-5-v)), &\quad \text{for } i=0 \text{ and } 2 \\ h^i(\mathcal{O}_{X'}(v)) = h^{2-i}(\mathcal{I}_{X/Y}(f+g-5-v)), &\quad \text{for } i=0 \text{ and } 2 \end{aligned} \tag{27}$$

from which we deduce $d+d' = fg$ and $\pi' - \pi = (d'-d)(f+g-4)/2$.

The generalization of (27) to equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ is clear, e.g., we have

$$h^i(\mathcal{I}_{X'/Y}(v)) = h^{n-i}(\mathcal{O}_X(f+g-n-3-v)), \quad \text{for } i=0 \text{ and } n. \tag{28}$$

Note that we now have n deficiency modules, whose dimensions $\rho_i(v) = h^i(\mathcal{I}_X(v))$, $i = 1,2,\ldots,n$ determine the vector function $\rho = (\rho_1,\ldots,\rho_n)$. Using this vector function, we easily generalize (25) in such a way that we get the tangent space ${}_0\mathrm{Ext}^1_R(I_X,I_X)_\rho$ of the Hilbert scheme $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ of constant cohomology in this case. We allow $n=0$ in which case there is no ρ and $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P}^2)$ should be taken as the Hilbert scheme of constant postulation ("the postulation Hilbert scheme") and ${}_0\mathrm{Ext}^1_R(I_X,I_X)_\rho$ as ${}_0\mathrm{Ext}^1_R(I_X,I_X)$. We have (cf. [38] for the curve case of the theorem),

Theorem 7.1. *Let X and X' be two equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, linked by a complete intersection $Y \subseteq \mathbb{P}^{n+2}$ of type (f,g), and suppose that (X) (resp. (X')) belongs to the Hilbert scheme $\mathrm{H}_{\gamma,\rho}$ (resp. $\mathrm{H}_{\gamma',\rho'}$) of constant cohomology. Then*

i) $$\begin{aligned} &\dim_{(X)} \mathrm{H}_{\gamma,\rho} + h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)) \\ &\quad = \dim_{(X')} \mathrm{H}_{\gamma',\rho'} + h^0(\mathcal{I}_{X'}(f)) + h^0(\mathcal{I}_{X'}(g)) \end{aligned}$$

or equivalently,

$$\begin{aligned} \dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} &+ h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ &- h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)). \end{aligned}$$

ii) *The dimension formulas of* i) *remain true if we replace* $\dim_{(X)} \mathrm{H}_{\gamma,\rho}$ *and* $\dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ *by the dimensions of their tangent spaces* ${}_0\mathrm{Ext}^1_R(I_X,I_X)_\rho$ *and* ${}_0\mathrm{Ext}^1_R(I_{X'},I_{X'})_{\rho'}$ *respectively.*

iii) $\mathrm{H}_{\gamma,\rho}$ *is smooth at* (X) *if and only if* $\mathrm{H}_{\gamma',\rho'}$ *is smooth at* (X')

Proof. Let $D(p(v);f,g)$ be the Hilbert flag scheme parametrizing pairs (X,Y) of equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ such that Y is a CI of type (f,g) containing X. By [29], Thm. 2.6, there is an isomorphism of schemes,

$$D(p(v);f,g) \simeq D(p'(v);f,g), \tag{29}$$

given by sending (X, Y) onto (X', Y) where X' is linked to X by Y. We may suppose $n \geq 1$ in Theorem 7.1 since the case $n = 0$ is completely solved by Prop. 1.7 of [32]. Then the projection morphism $p : D(p(v); f, g) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$, given by $(X, Y) \mapsto (X)$, is smooth at (X, Y) provided $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ ([29], Thm. 1.16 (b)). By [29], Lem. 1.17 and Rem. 1.20, this smoothness holds if we replace the vanishing above with the claim that the set of global sections of the corresponding twisted ideal sheaves over the local ring of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) are locally free and commute with base change. Hence the following restriction of p to $p^{-1}(\mathrm{H}_{\gamma,\rho})$, $p^{-1}(\mathrm{H}_{\gamma,\rho}) \to \mathrm{H}_{\gamma,\rho}$, is smooth, (or see [38] for related arguments). Since the fiber dimension of p at (X, Y) is precisely

$$h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) = h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)) - h^0(\mathcal{I}_Y(f)) - h^0(\mathcal{I}_Y(g))$$

by [29], Thm. 1.16 (a), we get any conclusion of the theorem if we combine with (28). □

Remark 7.2. Let X and X' be two surfaces in $\mathbb{P}^4$, linked by a CI of type (f, g). Then the arguments of the proof above show that we can, under the assumptions

$$H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0 \quad \text{and} \quad H^1(\mathcal{I}_{X'}(f)) = H^1(\mathcal{I}_{X'}(g)) = 0 \qquad (30)$$

replace $\mathrm{H}_{\gamma,\rho}$ and $\mathrm{H}_{\gamma',\rho'}$ in Theorem 7.1 (i) (resp. their tangent spaces in Theorem 7.1 (ii)) by $\mathrm{H}(d, p, \pi)$ and $\mathrm{H}(d', p', \pi')$ (resp. by $H^1(\mathcal{N}_X)$ and $H^1(\mathcal{N}_{X'})$) and get valid dimension formulas involving the whole Hilbert schemes (resp. their tangent spaces). Hence assuming (30), it follows that X is unobstructed if and only if X' is unobstructed, see [29], Prop. 3.12 for a generalization. Note also that we from the proof above (i.e., from [29], Thm. 1.16 (b)) and (30) get that X is generic if and only if X' is generic, see [29], Prop. 3.8 for a related general result.

Example 7.3. Let X be the smooth rational surface of $\mathrm{H}(11, 0, 11)$ of Example 4.6, let Y be a CI of type $(5, 5)$ containing X, and let X' be the linked surface. Using (27) we deduce $\chi(\mathcal{O}_{X'}(v)) = 7v^2 - 12v + 9$ from $\chi(\mathcal{O}_X(v)) = (11v^2 - 9v + 2)/2$, i.e., (X') belongs to $\mathrm{H}(d', p', \pi') = \mathrm{H}(14, 8, 20)$ by (1). Moreover $\omega_{X'} = \mathcal{I}_{X/Y}(5)$ is globally generated (cf. the resolution of I of Example 4.6) and the graded modules $M'_i \simeq \oplus H^i(\mathcal{I}_{X'}(v))$ are supported at two consecutive degrees and satisfy

$$\dim H^1(\mathcal{I}_{X'}(3)) = 1, \qquad \dim H^2(\mathcal{I}_{X'}(1)) = 1,$$
$$\dim H^1(\mathcal{I}_{X'}(4)) = 3, \qquad \dim H^2(\mathcal{I}_{X'}(2)) = 2.$$

From these informations we find the minimal resolution of $I' = I_{X'}$ to be

$$0 \to R(-9)^{\oplus 3} \to R(-8)^{\oplus 14} \to R(-7)^{\oplus 23} \to R(-6)^{\oplus 11} \oplus R(-5)^{\oplus 2} \to I' \to 0.$$

Combining Example 4.6 and Remark 6.2 we see that $\mathrm{H}_{\gamma,\rho}$ is smooth at (X) and $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = 41$. Thanks to Theorem 7.1, we get that $\mathrm{H}_{\gamma',\rho'}$ is smooth at (X') and that

$$\dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} + 2h^0(\mathcal{I}_{X/Y}(5)) - 2h^2(\mathcal{O}_X(0)) = 57.$$

Moreover by Remark 7.2 or Proposition 6.1, $\mathrm{H}(d',p',\pi') \simeq \mathrm{H}_{\gamma',\rho'}$ is smooth at (X') and $\dim_{(X')} \mathrm{H}(d',p',\pi') = 57$. Note that in this case we neither have

$$ {}_0\mathrm{Ext}^3(M_2,M_1) = 0 \quad \text{nor} \quad {}_{-5}\mathrm{Hom}_R(I,M_2) = 0, $$

i.e., we can not use Corollary 5.4 or Proposition 4.4 to conclude that $\mathrm{H}_{\gamma',\rho'}$ is smooth at (X'). But, as we have seen, the linkage result above takes care of the smoothness and the dimension.

If a surface X of $\mathbb{P}^4$ is contained in a CI Y of type (f,g), then there is an inclusion map $I_Y \to I_X$ which induces a morphism $l^{i+1}_{X/Y} : H^i(\mathcal{N}_X) \to H^i(\mathcal{O}_X(f)) \oplus H^i(\mathcal{O}_X(g))$ for every i. We let $\beta_{X/Y}$ be the composition of $l^1_{X/Y}$ with the natural map $H^0(\mathcal{O}_X(f)) \oplus H^0(\mathcal{O}_X(g)) \to H^1(\mathcal{I}_X(f)) \oplus H^1(\mathcal{I}_X(g))$.

Proposition 7.4. *Let X and X' be surfaces in $\mathbb{P}^4$, geometrically linked by a complete intersection $Y \subseteq \mathbb{P}^4$ of type (f,g), let $(X) \in \mathrm{H}_{\gamma,\rho}$ and $(X') \in \mathrm{H}_{\gamma',\rho'}$ and suppose $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim_{(X)} \mathrm{H}(d,p,\pi)$. Let $c := \dim_{(X')} \mathrm{H}(d',p',\pi') - \dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ and suppose $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ and that $l^2_{X/Y}$ is injective. Then*

$$ h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)) - h^2(\mathcal{I}_{X'}(f)) - h^2(\mathcal{I}_{X'}(g)) \le c \le h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)) \tag{31} $$

and we have equality on the right-hand side if and only if $\mathrm{H}(d',p',\pi')$ is smooth at (X'). Furthermore, if $h^1(\mathcal{I}_{X'}(v)) \cdot h^2(\mathcal{I}_{X'}(v)) = 0$ for $v = f$ and $v = g$, then

$$ c = h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)). $$

Proof. Since X and X' are generically complete intersections (due to geometric linkage) of codimension 2 in $\mathbb{P}^4$, it follows that the cotangent sheaves $\mathcal{A}^2_X$ and $\mathcal{A}^2_{X'}$ are zero (cf. [10]). The vanishing of the obstruction group, $A^2(X \subseteq Y)$, of the Hilbert flag scheme $D(p(v);f,g)$ at (X,Y) is therefore equivalent to $\beta_{X/Y}$ being surjective and $l^2_{X/Y}$ being injective by (1.11) of [29], so $A^2(X \subseteq Y) = 0$ by assumption. Moreover since the linkage is geometric, we get $A^2(X' \subseteq Y) = 0$ by Cor. 2.14 of [29], i.e., $\beta_{X'/Y}$ is surjective, $l^2_{X'/Y}$ is injective and $D(p'(v);f,g)$ is smooth at (X',Y). Hence [29], Thm. 1.27 applies (to a component V satisfying $\dim V = \dim_{(X')} \mathrm{H}(d',p',\pi')$) to get the bounds of the codimension c above provided we can show that $\mathrm{H}_{\gamma',\rho'}$, in a neighborhood of (X'), is dense in an (f,g)-maximal subset of $\mathrm{H}(d',p',\pi')$ (i.e., is dense in the image under the first projection of some non-embedded component of $D(p'(v);f,g)$). By the proof of Theorem 7.1 we see that the restriction of the first projection p' to $p'^{-1}(\mathrm{H}_{\gamma',\rho'})$, $p'^{-1}(\mathrm{H}_{\gamma',\rho'}) \to \mathrm{H}_{\gamma',\rho'}$, is smooth. It follows that $\mathrm{H}_{\gamma',\rho'}$ is, locally at (X'), (f,g)-maximal provided we can show

$$ \dim_{(X',Y)} p'^{-1}(\mathrm{H}_{\gamma',\rho'}) = \dim_{(X',Y)} D(p'(v);f,g). $$

Thanks to (29) it suffices to show $\dim_{(X,Y)} p^{-1}(\mathrm{H}_{\gamma,\rho}) = \dim_{(X,Y)} D(p(v);f,g)$ which readily follows from the assumptions $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim_{(X)} \mathrm{H}(d,p,\pi)$ and $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ because the first projection, $p : D(p(v);f,g) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^4)$, as well as its restriction to $p^{-1}(\mathrm{H}_{\gamma,\rho})$, are smooth at (X,Y) by Re-

mark 7.2. Then we get the final conclusion from [29], Cor. 1.29, which states that $h^1(\mathcal{I}_{X'}(v)) \cdot h^2(\mathcal{I}_{X'}(v)) = 0$ for $v = f$ and g implies that $\mathrm{H}(d', p', \pi')$ is smooth at (X') and we are done. □

Example 7.5. Let Z be the surface which is linked to the surface $(X') \in \mathrm{H}(14, 8, 20)$ of Example 7.3 via a complete intersection of type $(5, 6)$ containing X'. Then (Z) belongs to $\mathrm{H}(16, 15, 27)$, $\omega_Z = \mathcal{I}_{X'/Y}(6)$ is globally generated, and $M_i(Z) = \oplus H^i(\mathcal{I}_Z(v))$, $i = 1, 2$, are supported at two consecutive degrees. Moreover;

$$\begin{array}{lll} h^0(\mathcal{I}_Z(5)) = 1, & h^1(\mathcal{I}_Z(4)) = 2 \quad \text{and} & h^1(\mathcal{I}_Z(5)) = 1 \\ h^2(\mathcal{O}_Z(1)) = 1, & h^2(\mathcal{I}_Z(2)) = 3 \quad \text{and} & h^2(\mathcal{I}_Z(3)) = 1\ . \end{array} \tag{32}$$

By Proposition 4.1, we know $\chi(\mathcal{N}_{X'}) = 5(2d' + \pi' - 1) - d'^2 + 2\chi(\mathcal{O}_{X'}) = 57$ and since we obviously have $h^2(\mathcal{N}_{X'}) = 0$ (from $h^2(\mathcal{O}_{X'}(1)) = 0$) and we get $h^0(\mathcal{N}_{X'}) = 57$ from Example 7.3, we conclude that $h^1(\mathcal{N}_{X'}) = 0$. The conditions of Proposition 7.4 are therefore satisfied (replacing X by X' there). Hence, at (Z), we get that $\mathrm{H}(16, 15, 27)_{\gamma,\rho}$ is smooth of codimension 1 in $\mathrm{H}(16, 15, 27)$. Moreover $\mathrm{H}(16, 15, 27)$ is smooth at (Z), and

$$\begin{aligned} &\dim_{(Z)} \mathrm{H}(16, 15, 27)_{\gamma,\rho} \\ &= \dim_{(X')} \mathrm{H}_{\gamma',\rho'} + h^0(\mathcal{I}_{X'/Y}(5)) + h^0(\mathcal{I}_{X'/Y}(6)) - h^2(\mathcal{O}_{X'}) - h^2(\mathcal{O}_{X'}(1)) = 65. \end{aligned}$$

Hence Z belongs to a unique generically smooth component V of $\mathrm{H}(16, 15, 27)$ of dimension 66, and since the generic surface $\tilde{Z}$ of V do not have the same cohomology as Z (since $\tilde{Z} \notin \mathrm{H}(16, 15, 27)_{\gamma,\rho}$), we must get

$$\dim H^0(\mathcal{I}_{\tilde{Z}}(5)) = \dim H^1(\mathcal{I}_{\tilde{Z}}(5)) = 0$$

while elsewhere the dimension of the cohomology groups is unchanged, i.e., it is as in (32).

8. Obstructed surfaces in ^{4}

In this section we explicitly prove the existence of obstructed surfaces. Our examples are as close as they can be to the arithmetically Cohen-Macaulay case. Indeed, in the examples, one of the Rao modules in the pair (M_1, M_2) vanishes, the other is 1-dimensional. Moreover in Proposition 4.4 and Remark 4.5 we gave conditions which imply unobstructedness. Our Example 8.3 is minimal with respect to the mentioned conditions in the sense that only one of the many cohomology groups, claimed in Remark 4.5 (i) to vanish, is non-zero. It also shows that we in Remark 7.2 can not skip the assumption (30) since we in Example 8.3 link an unobstructed surface to an obstructed surface where one of the cohomology groups of (30) is non-zero. Moreover, note that once having constructed one obstructed surface we can find infinitely many by linking under the assumption (30).

In the following proposition we consider a codimension 2 subscheme X of $\mathbb{P}^{n+2}$, containing a CI Y of type (f_1, f_2), in order to find obstructed codimension 2 subschemes of $\mathbb{P}^{n+2}$ for $n \geq 1$. In this situation we recall that the inclusion map

$I_Y \to I_X$ induces a morphism $H^0(\mathcal{N}_X) \to \oplus_{i=1}^2 H^0(\mathcal{O}_X(f_i))$ whose composition with $\oplus_{i=1}^2 H^0(\mathcal{O}_X(f_i)) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$ we denote $\beta_{X/Y}$. Note that we below do not need the cotangent sheaves to vanish since we work only with tangent (and not construction) spaces of $D(p(v); f_1, f_2)$.

Proposition 8.1. *Let X be an equidimensional locally Cohen-Macaulay codimension 2 subscheme of $\mathbb{P}^{n+2}$, and let Y and Y_0 be two complete intersections containing X, both of type (f_1, f_2) such that*

i) *$\beta_{X/Y}$ is surjective and β_{X/Y_0} is not surjective,*

ii) *$H^n(\mathcal{I}_X(f_i - n - 3)) = 0$ for $i = 1$ and $i = 2$.*

Let X' (resp. X_0') be linked to X by Y (resp. Y_0). Then X_0 is obstructed. Moreover if X is unobstructed, then so is X'.

Proof. If $A^1(X \subseteq Y)$ is the tangent space of the Hilbert flag scheme $D(p(v); f_1, f_2)$ at (X, Y), then it is shown in [29], (1.11) that there is an exact sequence

$$0 \to \oplus_{i=1}^2 H^0(\mathcal{I}_{X/Y}(f_i)) \to A^1(X \subseteq Y) \to H^0(\mathcal{N}_X) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$$

where the rightmost map is $\beta_{X/Y}$. The corresponding exact sequence for $(X \subseteq Y_0)$ together with the assumption (i) show that

$$\dim A^1(X \subseteq Y) < \dim A^1(X \subseteq Y_0)$$

because it is easy to see $h^0(\mathcal{I}_{X/Y}(v)) = h^0(\mathcal{I}_{X/Y_0}(v))$ for every v. We *claim* that $D(p(v); f_1, f_2)$ is not smooth at (X, Y_0). Suppose the converse. Since it is shown in [29], Thm. 1.16 (a) that the fibers of the first projection $p : D(p(v); f_1, f_2) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ are irreducible, it follows that there exists an irreducible component W of $D(p(v); f_1, f_2)$ which contains both points, (X, Y) and (X, Y_0). Hence if $D(p(v); f_1, f_2)$ is smooth at (X, Y_0), we get

$$\dim A^1(X \subseteq Y_0) = \dim W \leq \dim_{(X,Y)} D(p(v); f_1, f_2) \leq \dim A^1(X \subseteq Y),$$

i.e., a contradiction.

Thanks to (29) we get that $D(p'(v); f_1, f_2)$ is not smooth at (X_0', Y_0). Since $h^1(\mathcal{I}_{X_0'}(f_i - n - 3)) = h^n(\mathcal{I}_X(f_{3-i} - n - 3)) = 0$ for $i = 1, 2$, cf. (27), and since the vanishing of $H^1(\mathcal{I}_{X_0'}(f_i - n - 3))$ implies that the first projection $p' : D(p'(v); f_1, f_2) \to \mathrm{Hilb}^{p'(v)}(\mathbb{P}^{n+2})$ is smooth at (X_0', Y_0) by [29], Thm. 1.16 (b), we conclude that X_0' is obstructed. Finally, for the last conclusion, if we have the surjectivity of $\beta_{X/Y}$ and assume the unobstructedness of X, we get that $D(p(v); f_1, f_2)$ is smooth at (X, Y) by [29], Prop. 3.12. Using (29) and (27) once more we conclude that X' is unobstructed, and we are done. □

We think the surjectivity of $\beta_{X/Y}$ may often hold, provided the generators of I_Y are among the minimal generators of I_X, but this is difficult to prove. In the Buchsbaum case, however, it is easy to see the surjectivity, as observed in [7] for curves. Indeed even though the statement of Proposition 8.1 and the remark below generalizes [7], Prop. 2.1 by far, the ideas of the proof are quite close to the idea in Prop. 2.1 of [7].

Remark 8.2. In this remark we consider surfaces in $\mathbb{P}^4$ with minimal resolution given as in (10).

(i) Using (5) and the spectral sequence (2) we get an exact sequence

$$\to H^0(\mathcal{N}_X) \to {}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \xrightarrow{\alpha} {}_0\mathrm{Ext}^2_R(I_X, I_X) \to$$

where ${}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \simeq \oplus_i H^1(\mathcal{I}_X(n_{1,i}))$ provided $H^1(\mathcal{I}_X(n_{2,i})) = 0$ for any i. The natural map

$$H^0(\mathcal{N}_X) \to {}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \simeq \oplus_i H^1(\mathcal{I}_X(n_{1,i})),$$

which we denote β_X, is correspondingly defined as $\beta_{X/Y}$ above, but with the difference that a set of all minimal generators of I_X is used. In particular if the generators of I_Y are among the minimal generators of I_X, then the composition of β_X with the projection $\oplus_i H^1(\mathcal{I}_X(n_{1,i})) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$ is $\beta_{X/Y}$. It follows that if

$${}_0\mathrm{Ext}^2_R(I_X, I_X) = 0 \quad \text{and} \quad H^1(\mathcal{I}_X(n_{2,i})) = 0 \text{ for any i ,}$$

then $\beta_{X/Y}$ is surjective. Note that, by (3) and (2) (cf. the proof of Proposition 4.4), ${}_0\mathrm{Ext}^2_R(I_X, I_X) = 0$ provided ${}_{-5}\mathrm{Ext}^1_R(I, M_1) = {}_{-5}\mathrm{Hom}_R(I, M_2) = 0$, i.e., provided

$$H^1(\mathcal{I}_X(n_{2,i} - 5)) = 0 \text{ and } H^2(\mathcal{I}_X(n_{1,i} - 5)) = 0 \text{ for every } i.$$

(ii) If, however, the minimal generators $\{F_1, F_2\}$ of I_Y do not belong to a set of minimal generators of I_X, say $F_i = H_i \cdot G_i$ for some $G_i \in I_X$, $i = 1, 2$, then $\beta_{X/Y}$ is easily seen to be non-surjective under a manageable assumption. Indeed let g_i be the degree of the form G_i, let Y_0 be the CI with homogeneous ideal $I_{Y_0} = (G_1, G_2)$ and suppose the obvious map

$$h \ : \ \oplus_{i=1}^2 H^1(\mathcal{I}_X(g_i)) \xrightarrow{(H_1, H_2)} \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i)) \ \text{ is not surjective.}$$

Then $\beta_{X/Y}$ can not be surjective because it factors via h, i.e., $\beta_{X/Y} = h \circ \beta_{X/Y_0}$!

Example 8.3. If we link the smooth quintic scroll Z of $\mathrm{H}(5, -1, 1)$ with Rao modules $H^1_*(\mathcal{I}_Z) = 0$, $H^2_*(\mathcal{I}_Z) \simeq k$ and minimal resolution (cf. [11], B.2.1),

$$0 \to R(-5) \to R(-4)^{\oplus 5} \to R(-3)^{\oplus 5} \to I_Z \to 0, \tag{33}$$

using a CI of type $(5, 6)$ containing Z, then the ideal of the linked surface X has a minimal resolution

$$0 \to R(-11) \to R(-10)^{\oplus 5} \to R(-9)^{\oplus 10} \to R(-8)^{\oplus 5} \oplus R(-6) \oplus R(-5) \to I_X \to 0$$

and Rao modules given by $H^2_*(\mathcal{I}_X) = 0$, $h^1(\mathcal{I}_X(6)) = 1$ and $H^1(\mathcal{I}_X(v)) = 0$ for $v \neq 6$. Using (27) we see that (X) belongs to $\mathrm{H}(d, p, \pi) = \mathrm{H}(25, 99, 71)$. This surface X has invariants such that Proposition 8.1 and Remark 8.2 apply. Indeed we can link X to two different surfaces X' and X'_0 using CI's Y and Y_0 containing X, both of type $(6, 8)$, generated in the following way. Let F_5, resp. F_6, be the minimal generator of I_X of degree 5, resp. 6, and let G be a general element

of $H^0(\mathcal{I}_X(8))$. Then we take Y, resp. Y_0, to be given by $I_Y = (F_6, G)$, resp. $I_{Y_0} = (H \cdot F_5, G)$ where H is a linear form. We may check that all assumptions of Remark 8.2 are satisfied. Hence we get that X' and X'_0 belong to a common irreducible component of $\mathrm{H}(d', p', \pi') = \mathrm{H}(23, 80, 61)$, that X'_0 is obstructed with minimal resolution

$$0 \to R(-8) \to R(-7)^{\oplus 5} \oplus R(-8) \oplus R(-9) \to R(-6)^{\oplus 6} \oplus R(-8) \to I_{X'_0} \to 0,$$

while X' is unobstructed with minimal resolution

$$0 \to R(-8) \to R(-7)^{\oplus 5} \oplus R(-9) \to R(-6)^{\oplus 6} \to I_{X'} \to 0.$$

Note that it is straightforward to find these resolutions since X' and X'_0 are bilinked to Z and we know the minimal resolution of I_Z, see [39] or the sequence (39) appearing later in this paper. We observe that common direct free factors ("ghost terms") are present in the minimal resolution, similar to what happens for obstructed curve with "small Rao module", cf. [33]. Moreover since the assumptions of Proposition 4.4 are satisfied for X', we also get the unobstructedness of X' from that Proposition and the dimension,

$$\dim_{(X')} \mathrm{H}(23, 80, 61) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) = 163.$$

However, since the conditions of Remark 4.5 (i) also hold, we get $H^1(\mathcal{N}_{X'}) = 0$ and hence it is easier to compute $\dim_{(X')} \mathrm{H}(23, 80, 61)$ by using Proposition 4.1. We get

$$\dim_{(X')} \mathrm{H}(23, 80, 61) = \chi(\mathcal{N}_{X'}) = 5(2d' + \pi' - 1) - d'^2 + 2\chi(\mathcal{O}_{X'}) = 163.$$

Note that neither the assumptions of Proposition 4.4, nor the assumptions of Remark 4.5 (i), are satisfied for X'_0. Indeed Remark 4.5 (i) a little extended will show $h^1(\mathcal{N}_{X'_0}) = 1$ (i.e., just compute the dimension using (17)). The surface X'_0 is easily seen to be reducible, as pointed out to me by H. Nasu.

Example 8.4. If we link the surface X'_0 of Example 8.3 using a general CI of type $(9, 9)$ containing X'_0, we get a smooth obstructed surface S of degree 58. Indeed the assumptions of Remark 7.2 are satisfied. So S is obstructed, and we have used Macaulay 2 ([19]) to verify that S is smooth provided the CI's used in the linkages of Example 8.3 are general enough under the specified restrictions. The surface S is in the biliaison class of the Veronese surface in $\mathbb{P}^4$.

Finally if we link S via a general CI of type $(9, 12)$ containing S, we get an obstructed surface S' of degree 50 by Remark 7.2. We have used Macaulay 2 to verify that the surface is smooth. The surface S' is in the biliaison class of the quintic elliptic scroll in $\mathbb{P}^4$. Since S' is bilinked to the surface X'_0 of Example 8.3 we easily find the minimal resolution of $I_{S'}$ to be

$$0 \to R(-11) \to R(-10)^{\oplus 5} \oplus R(-11) \oplus R(-12)^{\oplus 2} \to R(-9)^{\oplus 7} \oplus R(-11) \to I_{S'} \to 0.$$

Note that we again have "ghost terms" in the minimal resolution in degree $c + 5$ where $h^2(\mathcal{I}_{S'}(c)) \neq 0$. This feature seems to be related to obstructedness, as in the curve case, cf. [33].

9. Even liaison of codimension 2 subschemes of $^{n+2}$

In this section we prove the main even liaison theorem of this paper, which holds for any equidimensional lCM codimension 2 subscheme X of $\mathbb{P}^{n+2}$. We also generalize Proposition 4.4 and the vanishing result for $h^1(\mathcal{N}_X)$ of Remark 4.5 to schemes X of dimension $n > 2$ and we give an example of an obstructed 3-fold.

First we define $\delta^m_X(v)$. Let

$$0 \to \bigoplus_{i=1}^{r_{n+2}} R(-n_{n+2,i}) \to \bigoplus_{i=1}^{r_{n+1}} R(-n_{n+1,i}) \to \cdots$$
$$\cdots \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0 \qquad (34)$$

be a minimal resolution of $I = I_X$ and let the invariant $\delta^m(v) = \delta^m_X(v)$ be defined by

$$\delta^m_X(v) = \sum_{j=1}^{n+2} \sum_{i=1}^{r_j} (-1)^{j+1} h^m(\mathcal{I}_X(n_{j,i} + v)) \ . \qquad (35)$$

Since adding common direct free factors in consecutive terms of (34) does not change $\delta^m_X(v)$, the resolution of I does not really need to be minimal in the definition of $\delta^m_X(v)$.

Theorem 9.1. *Let X and X' be two equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, linked to each other in two steps by two complete intersections, and suppose that (X) (resp. (X')) belongs to the Hilbert scheme $\mathrm{H}_{\gamma,\rho}$ (resp. $\mathrm{H}_{\gamma',\rho'}$) of constant cohomology. Then*

i) $\delta^{n+1}_X(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} = \delta^{n+1}_{X'}(-n-3) - \dim_{(X')} \mathrm{H}_{\gamma',\rho'}$.

In particular

$$\mathrm{obsumext}(X) := 1 + \delta^{n+1}_X(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho}$$

is a biliaison invariant.

ii) $\delta^{n+1}_X(-n-3) - \dim\ {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho = \delta^{n+1}_{X'}(-n-3) - \dim\ {}_0\mathrm{Ext}^1_R(I_{X'}, I_{X'})_{\rho'}$.

In particular

$$\mathrm{sumext}(X) := 1 + \delta^{n+1}_X(-n-3) - \dim\ {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$$

is a biliaison invariant.

iii) *We have* $\mathrm{sumext}(X) \le \mathrm{obsumext}(X)$, *with equality if and only if* $\mathrm{H}_{\gamma,\rho}$ *is smooth at* (X).

Remark 9.2. This result is motivated by Remarks 3.9 and 6.4. Indeed we were quite convinced that Theorem 9.1 was true before starting proving it. Note that the dimension formula of Remark 6.4 was quite involved already for the case $n = \dim X = 2$ and we expect a very complicated formula for $n > 2$. So Theorem 9.1 may be a good practical approach to the problem of studying $\mathrm{H}_{\gamma,\rho}$ and

$\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ with respect to smoothness and dimension for $n > 1$. However, except for the other results of this paper, we have no better option for the use of Theorem 9.1 that to first compute $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ through a nice representative in the even liaison class, e.g., for the minimal element of the class, before we use it for an arbitrary element in the even liaison class.

Remark 9.3. For the application of Theorem 9.1 there is one natural situation where $\mathrm{H}_{\gamma,\rho}$ is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X), namely in the case X has seminatural cohomology. We say a subscheme $X \subseteq \mathbb{P}^{n+2}$ has seminatural cohomology if for every $v \in \mathbb{Z}$, at most one of groups $H^0(\mathcal{I}_X(v)), H^1(\mathcal{I}_X(v)), \ldots, H^{n+1}(\mathcal{I}_X(v))$ are non-zero. In this case a generization (i.e., a deformation to more general element in $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$) of X is forced to have the same cohomology as X by the semicontinuity of $h^i(\mathcal{I}_X(v))$, i.e., $\mathrm{H}_{\gamma,\rho} \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ as schemes at (X).

Proof. Let X be linked to X_1 by a CI $Y \subseteq \mathbb{P}^{n+2}$ of type (f, g) and let X_1 be linked to X' by some CI $Y' \subseteq \mathbb{P}^{n+2}$ of type (f', g'). If (X_1) belongs to the Hilbert scheme $\mathrm{H}_1 := \mathrm{H}_{\gamma_1,\rho_1}$ of constant cohomology, then by Theorem 7.1,

$$\begin{aligned}\dim_{(X_1)} \mathrm{H}_1 = \dim_{(X)} \mathrm{H}_{\gamma,\rho} &+ h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ &- h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)),\end{aligned}$$

$$\begin{aligned}\dim_{(X_1)} \mathrm{H}_1 = \dim_{(X')} \mathrm{H}_{\gamma',\rho'} &+ h^0(\mathcal{I}_{X'/Y'}(f')) + h^0(\mathcal{I}_{X'/Y'}(g')) \\ &- h^n(\mathcal{O}_{X'}(f'-n-3)) - h^n(\mathcal{O}_{X'}(g'-n-3)).\end{aligned}$$

Let $h = f' + g' - f - g$. Using (28) twice we get $h^0(\mathcal{I}_{X'/Y'}(v)) = h^0(\mathcal{I}_{X/Y}(v-h))$. Hence

$$\begin{aligned}\dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} &+ h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ &- h^0(\mathcal{I}_{X/Y}(f'-h)) + h^0(\mathcal{I}_{X/Y}(g'-h)) + \eta\end{aligned} \tag{36}$$

where η is defined by

$$\eta := h^n(\mathcal{O}_{X'}(f'-n-3)) + h^n(\mathcal{O}_{X'}(g'-n-3)) - h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)). \tag{37}$$

Next we need to find a free resolution of $I' = I_{X'}$ in terms of the minimal resolution of $I = I_X$ in (34). If we define E by the exact sequence

$$0 \to \oplus_{i=1}^{r_{n+2}} R(-n_{n+2,i}) \to \cdots \to \oplus_{i=1}^{r_3} R(-n_{3,i}) \to \oplus_{i=1}^{r_2} R(-n_{2,i}) \to E \to 0, \tag{38}$$

we may put (34) in the form $0 \to E \to \oplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0$. Then it is well known that there is an exact sequence

$$0 \to E(-h) \oplus R(-f-h) \oplus R(-g-h) \to \oplus_{i=1}^{r_1} R(-n_{1,i}-h) \oplus R(-f') \oplus R(-g') \to I' \to 0 \tag{39}$$

which combined with (38) yields a free resolution of I' (see [39]).

We will use this resolution of I' and (34) to see the connection between $\delta_X^{n+1}(-n-3)$ and $\delta_{X'}^{n+1}(-n-3)$. First we need to compute β defined by

$$\beta := \sum_{j=1}^{n+2} \sum_{i=1}^{r_j} (-1)^{j+1} \alpha(n_{j,i} - n - 3) \quad \text{where} \quad \alpha(v) := h^n(\mathcal{O}_{X'}(v+h)) - h^n(\mathcal{O}_X(v)) \ .$$

We *claim* that

$$\beta = h^0(\mathcal{I}_X(f))+h^0(\mathcal{I}_X(g))-h^0(\mathcal{I}_X(f'-h))-h^0(\mathcal{I}_X(g'-h))+h^0(\mathcal{I}_X(-h)). \quad (40)$$

Indeed by (28),

$$\alpha(v) = h^0(\mathcal{I}_{X_1/Y'}(f'+g'-n-3-v-h)) - h^0(\mathcal{I}_{X_1/Y}(f+g-n-3-v)).$$

Moreover since $0 \to \mathcal{I}_{Y'} \to \mathcal{I}_{X_1} \to \mathcal{I}_{X_1/Y'} \to 0$ and $0 \to \mathcal{I}_Y \to \mathcal{I}_{X_1} \to \mathcal{I}_{X_1/Y} \to 0$ are exact, we get

$$\alpha(v) = h^0(\mathcal{I}_Y(f+g-n-3-v)) - h^0(\mathcal{I}_{Y'}(f+g-n-3-v)). \quad (41)$$

Let $r(v) := \dim R_{(-n-3+v)}$. Combining with the minimal resolutions of I_Y and I'_Y, we get

$$\alpha(v) := r(f-v)+r(g-v)-r(-v)-r(f'-h-v)-r(g'-h-v)+r(-h-v).$$

Then we get the claim since (34) implies

$$h^0(\mathcal{I}_X(v)) = \sum_{j=1}^{n+2}\sum_{i=1}^{r_j}(-1)^{j+1}r(v-n_{j,i}+n+3)$$

for any v and since $h^0(\mathcal{I}_X(0)) = 0$.

Using the resolution of I' deduced from (39) and the definition (35) we get

$$\delta_{X'}^{n+1}(-n-3) = \sum_{j=1}^{n+2}\sum_{i=1}^{r_j}(-1)^{j+1}h^n(\mathcal{O}_{X'}(n_{j,i}+h-n-3)) + \epsilon$$

where ϵ is defined by

$$\begin{aligned}\epsilon := {} & h^n(\mathcal{O}_{X'}(f'-n-3)) + h^n(\mathcal{O}_{X'}(g'-n-3)) \\ & - h^n(\mathcal{O}_{X'}(f+h-n-3)) - h^n(\mathcal{O}_{X'}(g+h-n-3)).\end{aligned}$$

Comparing ϵ with η in (37) and recalling the definition of α, we have $\epsilon = \eta - \alpha(f-n-3) - \alpha(g-n-3)$. Moreover the definition of α, the proven claim and (35) lead to $\delta_{X'}^{n+1}(-n-3) = \delta_X^{n+1}(-n-3) + \beta + \epsilon$. Combining we get

$$\delta_{X'}^{n+1}(-n-3) = \delta_X^{n+1}(-n-3) + \beta + \eta - \alpha(f-n-3) - \alpha(g-n-3).$$

Comparing with (36) we get (i) of the Theorem provided we can show that

$$\begin{aligned}h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) - h^0(\mathcal{I}_{X/Y}(f'-h)) - h^0(\mathcal{I}_{X/Y}(g'-h)) \\ = \beta - \alpha(f-n-3) - \alpha(g-n-3).\end{aligned}$$

Suppose $h \geq 0$. Looking at (40), we see it suffices to show

$$-h^0(\mathcal{I}_Y(f))-h^0(\mathcal{I}_Y(g))+h^0(\mathcal{I}_Y(f'-h))+h^0(\mathcal{I}_Y(g'-h)) = -\alpha(f-n-3)-\alpha(g-n-3).$$

Thanks to (41) it remains to show

$$h^0(\mathcal{I}_Y(f'-h)) + h^0(\mathcal{I}_Y(g'-h)) = h^0(\mathcal{I}_{Y'}(f)) + h^0(\mathcal{I}_{Y'}(g)).$$

Using the minimal resolutions of I_Y and $I_{Y'}$ and that $h = f' + g' - f - g \geq 0$, we easily show that both sides of the last equation is equal to $\dim R_{(f-f')} + \dim R_{(f-g')} + \dim R_{(g-f')} + \dim R_{(g-g')}$ and we get what we want, i.e.,

$$\delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} = \delta_{X'}^{n+1}(-n-3) - \dim_{(X')} \mathrm{H}_{\gamma',\rho'} \tag{42}$$

provided $h \geq 0$. Suppose $h < 0$. Then we can start with X' and link in two steps back to X, i.e., we get an even liaison with $h' = f + g - f' - g' \geq 0$ in which case we know that (42) holds. Hence (42) is proved in general.

To show (ii) of the Theorem we only need to remark that, due to Theorem 7.1, (36) holds if we replace $\dim_{(X)} \mathrm{H}_{\gamma,\rho}$ and $\dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ by the dimension of their tangent spaces ${}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ and ${}_0\mathrm{Ext}^1_R(I_{X'}, I_{X'})_{\rho'}$ respectively. With the proof of Theorem 9.1 (i) above, we therefore get (42) with the mentioned replacements, i.e., we get Theorem 9.1 (ii).

Finally Theorem 9.1 (iii) follows by combining (i) and (ii) since, e.g., the smoothness of $\mathrm{H}_{\gamma,\rho}$ at (X) is equivalent to $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$. □

Corollary 9.4. *Let X be an equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$, and suppose (X) be a generic point of a generically smooth component V of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$. Then* $\mathrm{sumext}(X) = \mathrm{obsumext}(X)$ *and*

$$\dim V = 1 + \delta_X^{n+1}(-n-3) - \mathrm{sumext}(X).$$

Proof. Arguing as the last part of the proof of Theorem 3.7, we get that $\mathrm{H}_{\gamma,\rho}$ is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X). Hence $\mathrm{H}_{\gamma,\rho}$ is smooth at (X). Then we conclude by Theorem 9.1. □

Corollary 9.5. *Let X be a surface in $\mathbb{P}^4$. If the local deformation functors $Def(M_i)$ of M_i are formally smooth (for instance if ${}_0\mathrm{Ext}^2_R(M_i, M_i) = 0$) for $i = 1, 2$, and if ${}_0\mathrm{Ext}^3_R(M_2, M_1) = 0$, then*

$$\mathrm{sumext}(X) = \mathrm{obsumext}(X).$$

Proof. By Corollary 5.4 we get that $\mathrm{H}_{\gamma,\rho}$ is smooth at (X) and we conclude by Theorem 9.1 (iii). □

Corollary 9.6. *Let X be an arithmetically Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$. Then* $\mathrm{sumext(X)} = \mathrm{obsumext}(X) = 0$. *Moreover,*

(i) *if $n > 0$, then X is unobstructed and*

$$\begin{aligned}\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2}) &= 1 + \delta_X^{n+1}(-n-3) = 1 - \delta_X^0(0) \\ &= \chi(\mathcal{N}_X) + (-1)^n \delta_X^0(-n-3),\end{aligned}$$

(ii) *if $n = 0$, then H_γ is smooth at (X) and*

$$\dim_{(X)} \mathrm{H}_\gamma = 1 + \delta_X^1(-3) = 1 - \delta_X^0(0) = h^0(\mathcal{N}_X) + \delta_X^0(-3).$$

Proof. By Gaeta's theorem ([14], [15], cf. [2], [3]) X is in the liaison class of a complete intersection Y. Suppose $n > 0$. Then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) by [13] or [27], Rem. 3.7, (cf. [49], Thm. 2.1). Thanks to Theorem 9.1 it suffices to show that $\mathrm{sumext(Y)} = 0$, or equivalently that $\dim {}_0\mathrm{Ext}^1_R(I_Y, I_Y)_\rho = 1 + \delta_Y^{n+1}(-n-3)$. By definition, cf. (25), and (5), ${}_0\mathrm{Ext}^1_R(I_Y, I_Y)_\rho = {}_0\mathrm{Ext}^1_R(I_Y, I_Y) = h^0(\mathcal{N}_Y)$ and it is trivial to show $h^0(\mathcal{N}_Y) = 1 + \delta_Y^{n+1}(-n-3)$ by using duality and the minimal resolution of I_Y.

Moreover note that for *any* equidimensional lCM codimension 2 subschemes X of $\mathbb{P}^{n+2}$, we easily show

$$\sum_{i=1}^{n+1} {}_0\mathrm{ext}^i_R(I_X, I_X) = 1 - \delta^0_X(0) = \chi(\mathcal{N}_X) + (-1)^n \delta^0_X(-n-3). \tag{43}$$

as in Proposition 4.1 (see the first sentence of the proof for the left equality and second and third sentence of the proof for the right equality). Hence if X is arithmetically Cohen-Macaulay we get ${}_0\mathrm{ext}^i_R(I_X, I_X) = 0$ for $i \geq 2$ and we are done in the case $n > 0$. The case $n = 0$ is similar and easier. □

Remark 9.7. Corollary 9.6 coincides with [13] if $n > 0$, and with [18] and [36], Rem. 4.6 if $n = 0$.

Example 9.8. Let X be the smooth rational surface of $\mathrm{H}(11, 0, 11)$ of Example 4.6. Note that X has seminatural cohomology and hence we have $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d, p, \pi)$ at (X) by Remark 9.3. Moreover $I = I_X$ admits a minimal resolution

$$\begin{aligned} 0 \to R(-9) \to R(-8)^{\oplus 3} \oplus R(-7)^{\oplus 3} \to R(-7)^{\oplus 2} \oplus R(-6)^{\oplus 12} \\ \to R(-5)^{\oplus 10} \to I \to 0. \end{aligned} \tag{44}$$

By Example 4.6 we conclude that $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d, p, \pi)$ is smooth at (X) and that $\dim_{(X)} \mathrm{H}(d, p, \pi) = 41$. However, since X is rational we obviously get $1 + \delta^3_X(-5) = 1$ from (44). By Theorem 9.1 we find $\mathrm{sumext}(X) = \mathrm{obsumext}(X) = -40$. Now we link twice to get X', first using a CI of type $(5, 5)$, then a CI of type $(5, 6)$, both times using a common hypersurface of degree 5. Looking at (39) we find a free resolution of $I' = I_{X'}$ of the form

$$\begin{aligned} 0 \to R(-10) \to R(-9)^{\oplus 3} \oplus R(-8)^{\oplus 3} \to R(-8)^{\oplus 2} \oplus R(-7)^{\oplus 12} \oplus R(-6) \\ \to R(-6)^{\oplus 10} \oplus R(-5) \to I' \to 0. \end{aligned} \tag{45}$$

By (28), $h^2(\mathcal{O}_{X'}) = 15$ and $h^2(\mathcal{O}_{X'}(1)) = 1$ and we get $1 + \delta^3_{X'}(-5) = 25$. It follows from Theorem 9.1 and Proposition 6.1 that $\mathrm{H}_{\gamma',\rho'} \cong \mathrm{H}(d', p', \pi')$ is smooth at (X') of dimension $1 + \delta^3_{X'}(-5) - \mathrm{sumext}(X) = 65$. Compare with Examples 7.3 and 7.5.

Before considering examples of 3-folds, we want to generalize some of the results of Section 4. For recent papers on the Hilbert scheme of 3-folds, see [5] and its references. See also [12] for a long list of examples of 3-folds of non general type.

Proposition 9.9. *Let X be an equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$, let $M_i = H^i_*(\mathcal{I}_X)$ for $i = 1, \dots, n$ and $I = I_X$ and suppose*

$$ {}_0\mathrm{Hom}_R(I, M_1) = 0 \quad \text{and} \quad {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_j) = 0 \text{ for every } j,\ 1 \le j \le n. $$

Then ${}_0\mathrm{Ext}^2_R(I,I) = 0$, X is unobstructed and

$$ \dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2}) = {}_0\mathrm{ext}^1_R(I,I). $$

E.g., let $\dim X = 3$. *Then X is unobstructed and* $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = {}_0\mathrm{ext}^1_R(I,I)$ *if, for every i,*

$$ H^1(\mathcal{I}_X(n_{1,i})) = H^3(\mathcal{I}_X(n_{1,i}-6)) = H^2(\mathcal{I}_X(n_{2,i}-6)) = H^1(\mathcal{I}_X(n_{3,i}-6)) = 0. \tag{46} $$

If in addition

$$ H^2(\mathcal{I}_X(n_{1,i}-6)) = 0,\ H^1(\mathcal{I}_X(n_{2,i}-6)) = 0 \ \text{ and } \ H^1(\mathcal{I}_X(n_{1,i}-6)) = 0, $$

for every i, then $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta^0_X(0) = \chi(\mathcal{N}_X) - \delta^0_X(-6)$.

Proof. Thanks to [27], Rem. 3.7 (cf. [49], Thm. 2.1), the Hilbert scheme H_γ of constant postulation is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) provided ${}_0\mathrm{Hom}_R(I, M_1) = 0$. By (3) we get ${}_0\mathrm{Ext}^2_R(I,I) = 0$ provided ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$. By (2) and $M_j \cong H^{j+1}_{\mathfrak{m}}(I)$ we deduce the vanishing of the latter from the assumptions of the proposition. It follows that H_γ is smooth at (X) of dimension ${}_0\mathrm{ext}^1_R(I,I)$.

Suppose $n = 3$. By the definition of ${}_v\mathrm{Ext}^\bullet_R(I,-)$ and (34) we easily prove the vanishing of all $\mathrm{Ext}^\bullet_R(I,-)$-groups of the first part of the proposition from the explicit vanishing in (46). Moreover due (43), to get the final formula it suffices to show ${}_0\mathrm{Ext}^j_R(I,I) = 0$ for $j = 3, 4$. By (3) we must prove ${}_{-n-3}\mathrm{Ext}^{n-j}_{\mathfrak{m}}(I,I) = 0$ for $j = 0, 1$. This is shown in exactly the same way as we did for ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$, i.e., by using (2) and (34) and we are done. □

Remark 9.10.

(i) We can also generalize Remark 4.5 to equidimensional lCM codimension 2 subschemes $X \subseteq \mathbb{P}^{n+2}$ of higher dimension. Indeed using (5), (2) and (3), see the proof above, we get $H^1(\mathcal{N}_X) = 0$ provided ${}_0\mathrm{Ext}^3_{\mathfrak{m}}(I,I) = 0$ and ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$, e.g., provided

$$ {}_0\mathrm{Ext}^j_R(I, M_{2-j}) = 0 \text{ for } 0 \le j \le 1 \ \text{ and} $$
$$ {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_j) = 0 \text{ for } 1 \le j \le n. $$

Similarly $H^2(\mathcal{N}_X) = 0$ provided ${}_0\mathrm{Ext}^4_{\mathfrak{m}}(I,I) = 0$ and ${}_{-n-3}\mathrm{Ext}^n_{\mathfrak{m}}(I,I) = 0$, e.g., provided

$$ {}_0\mathrm{Ext}^j_R(I, M_{3-j}) = 0 \ \text{ for } 0 \le j \le 2 \ \text{ and} $$
$$ {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_{j-1}) = 0 \ \text{ for } 2 \le j \le n. $$

We can in this way easily get a vanishing criteria for $H^q(\mathcal{N}_X) = 0$ for every $q \ge 1$.

(ii) Suppose for instance $n = \dim X = 3$. Then $H^1(\mathcal{N}_X) = 0$ if, for every i,

$$H^1(\mathcal{I}_X(n_{2,i})) = H^2(\mathcal{I}_X(n_{2,i} - 6)) = 0,$$
$$H^2(\mathcal{I}_X(n_{1,i})) = H^3(\mathcal{I}_X(n_{1,i} - 6)) = 0 \text{ and } H^1(\mathcal{I}_X(n_{3,i} - 6)) = 0.$$

Moreover $H^2(\mathcal{N}_X) = 0$ if, for every i,

$$H^1(\mathcal{I}_X(n_{3,i})) = 0,\ H^2(\mathcal{I}_X(n_{2,i})) = H^1(\mathcal{I}_X(n_{2,i} - 6)) = 0 \text{ and}$$
$$H^3(\mathcal{I}_X(n_{1,i})) = H^2(\mathcal{I}_X(n_{1,i} - 6)) = 0.$$

As in the surface case, if some of the assumptions of Proposition 9.9 or Remark 9.10 are not satisfied, we can find examples of obstructed 3-folds (e.g., X_0' in the example below). Note that all assumptions of Proposition 9.9 and Remark 9.10 (ii) are satisfied for X_0', except $H^3(\mathcal{I}_{X_0'}(n_{1,i} - 6)) = 0$ for one i.

Example 9.11. We start with the smooth 3-fold $Z \subseteq \mathbb{P} := \mathbb{P}^5$ of [41] of degree 7 with Ω-resolution

$$0 \to \mathcal{O}^{\oplus 4} \to \Omega \ \ (2) \to \mathcal{I}_Z(4) \to 0,$$

where Ω is the kernel of the map $\mathcal{O}\ \ (-1)^6 \to \mathcal{O}$ induced by the multiplication with $(X_0, \ldots, X_5)$. Note that $h^1(\mathcal{I}_Z(2)) = 1$. If we link Z, first using a CI of type $(4, 4)$ to get a 3-fold Z', then a CI of type $(6, 7)$ to link Z' to X, then X is a 3-fold with properties such that Proposition 8.1 applies. Indeed the ideas of Remark 8.2 also apply except for how we proved ${}_0\mathrm{Ext}^2_R(I, I) = 0$. By the proof of Proposition 9.9, however, we have ${}_0\mathrm{Ext}^2_R(I, I) = 0$ for 3-folds provided $H^3(\mathcal{I}_X(n_{1,i} - 6)) = H^2(\mathcal{I}_X(n_{2,i} - 6)) = H^1(\mathcal{I}_X(n_{3,i} - 6)) = 0$ for all i. To see that all these $H^i(\mathcal{I}_X(j))$-groups vanish, we first find the minimal resolution of $I_{Z'}$. Combining the exact sequence $0 \to \mathcal{O}\ \ \to \mathcal{O}\ \ (1)^6 \to \Omega^\vee \to 0$ with the mapping cone construction for how we get the resolution of $I_{Z'}$ from the resolution of I_Z, we find the minimal resolution

$$0 \to R(-6) \to R(-5)^{\oplus 6} \to R(-4)^{\oplus 6} \to I_{Z'} \to 0.$$

Hence $H^1_*(\mathcal{I}_{Z'}) = 0$, $H^2_*(\mathcal{I}_{Z'}) = 0$ and we get $H^3_*(\mathcal{I}_X) = 0$, $H^2_*(\mathcal{I}_X) = 0$ and $H^1_*(\mathcal{I}_X) \simeq H^1(\mathcal{I}_X(7)) \simeq k$, cf. (27). Now since the Koszul resolution induced by the regular sequence $\{X_0, \ldots, X_5\}$ implies that

$$0 \to \mathcal{O}\ \ (-6) \to \mathcal{O}\ \ (-5)^{\oplus 6} \to \mathcal{O}\ \ (-4)^{\oplus 15} \to \mathcal{O}\ \ (-3)^{\oplus 20} \to \mathcal{O}\ \ (-2)^{\oplus 15} \to \Omega\ \ \to 0$$

is exact, we can use the mapping cone construction to find the following Ω-resolution,

$$0 \to \mathcal{O}\ \ (-9)^{\oplus 6} \to \Omega\ \ (-7) \oplus \mathcal{O}\ \ (-7) \oplus \mathcal{O}\ \ (-6) \to \mathcal{I}_X \to 0$$

of $\mathcal{I}_X$, leading to the minimal resolution

$$0 \to R(-13) \to R(-12)^{\oplus 6} \to R(-11)^{\oplus 15} \to \cdots \to I_X \to 0.$$

It follows that all $n_{3,i} = 11$ in the minimal resolution of I_X and hence we see that ${}_0\mathrm{Ext}^2_R(I, I) = 0$.

Then we proceed exactly as in Example 8.3. Indeed we link X to two different 3-folds X' and X_0' using CI's Y and Y_0 containing X, both of type $(7, 9)$, as follows.

Let F_6, resp. F_7, be the minimal generator of I_X of degree 6, resp. 7, and let G be a general element of $H^0(\mathcal{I}_X(9))$. Then we take Y, resp. Y_0, to be given by $I_Y = (F_7, G)$, resp. $I_{Y_0} = (H \cdot F_6, G)$ where H is a linear form. We may check that all assumptions of Proposition 8.1 are satisfied. Hence we get that X' and X'_0 belong to a common irreducible component of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$, that X'_0 is obstructed with minimal resolution

$$0 \to R(-9) \to R(-8)^{\oplus 6} \oplus R(-9) \oplus R(-10) \to R(-7)^{\oplus 7} \oplus R(-9) \to I_{X'_0} \to 0,$$

cf. (39), while X' is unobstructed with minimal resolution

$$0 \to R(-9) \to R(-8)^{\oplus 6} \oplus R(-10) \to R(-7)^{\oplus 7} \to I_{X'} \to 0.$$

Again we have "ghost terms" in the minimal resolution of $I_{X'_0}$. From the resolution we find X'_0 to be of degree 30 and with Hilbert polynomial

$$p(v) = 5v^3 - \frac{67}{2}v^2 + \frac{247}{2}v - 153.$$

The 3-fold X'_0 is reducible. Moreover since the assumptions of Proposition 9.9 are satisfied for X', we also get the unobstructedness of X' from that Proposition and the dimension, $\dim_{(X')} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta^0_{X'}(0) = 327$. Note that the assumptions of Proposition 9.9 are not satisfied for X'_0, due to the existence of a minimal generator of degree 9 of $I_{X'_0}$ and the fact $h^3(\mathcal{I}_{X'_0}(3)) = 1$.

Finally since Remark 7.2 generalizes to 3-folds by [29], Prop. 3.12, one may by linkage obtain infinitely many obstructed 3-folds in the liaison class of X'_0.

We will finish this section by finding the Hilbert polynomials of $\mathcal{O}_X$ and $\mathcal{N}_X$ for any equidimensional lCM 3-fold in $\mathbb{P}^5$ of degree d and sectional genus π. If S is a general hyperplane section, we have an exact sequence

$$0 \to \mathcal{O}_X(v-1) \to \mathcal{O}_X(v) \to \mathcal{O}_S(v) \to 0,$$

and we easily deduce

$$p(v) := \chi(\mathcal{O}_X(v)) = \frac{1}{6}dv^3 + \frac{1}{2}(d+1-\pi)v^2 + \left(\chi(\mathcal{O}_S) + \frac{d}{3} + \frac{1-\pi}{2}\right)v + \chi(\mathcal{O}_X) \quad (47)$$

from (1). Moreover

Proposition 9.12. *Let X be an equidimensional lCM 3-fold in $\mathbb{P}^5$ of degree d and sectional genus π and let S be a general hyperplane section. Then*

$$\chi(\mathcal{N}_X(v)) = \frac{1}{3}dv^3 + 3dv^2 + (2\chi(\mathcal{O}_S) + 5(\pi-1) + \frac{38}{3}d - d^2)v + (6\chi(\mathcal{O}_S) + 15(\pi-1) + 20d - 3d^2).$$

Proof. Since we have no reference for this formula in this generality we sketch a proof. Indeed we claim that

$$\chi(\mathcal{N}_X(v)) = \chi(\mathcal{O}_X(v)) - \chi(\mathcal{O}_X(-v-6)) - d^2(v+3). \quad (48)$$

Note that, using (48), we get Proposition 9.12 by combining with (47). To show (48), we follow the proof of Proposition 4.1. In addition to the formulas in (15)

(where we only replace $\sum_{j=1}^{4}$ by $\sum_{j=1}^{5}$) we get

$$\sum_{j=1}^{5}(-1)^{j-1}\sum_{i} n_{j,i}^{3} = 6(1-\pi-2d).$$

Then we proceed as in (16). We get $\delta^0(v) = -\chi(\mathcal{I}_X(-v-6)) - \chi(\mathcal{O}_X(v)) + (3+v)d^2$ for $v \gg 0$ and then the claim. □

Example 9.13. Let X be the smooth Calabi-Yau 3-fold of [12], Section 6, with invariants $d = 17$, $\pi = 32$, $\chi(\mathcal{O}_X) = 0$ and $\chi(\mathcal{O}_S) = 24$, and deficiency modules $M_1 = 0$, $M_2 = 0$ and M_3 given by

$$h^3(\mathcal{I}_X(1)) = 4, \quad h^3(\mathcal{I}_X(2)) = 2, \quad h^3(\mathcal{I}_X(v)) = 0 \ \text{ for } \ v \notin \{1,2\}.$$

Following [12] we find that $I = I_X$ has the following minimal resolution

$$0 \to R(-8)^{\oplus 2} \to R(-7)^{\oplus 8} \to R(-6)^{\oplus 5} \oplus R(-5)^{\oplus 2} \to I \to 0. \tag{49}$$

All assumptions of Proposition 9.9 are satisfied and we get that $\mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$ is smooth at (X) of dimension

$$\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta_X^0(0) = 82.$$

Let us compute obsumext(X). Note that X has seminatural cohomology and hence we have $\mathrm{H}_{\gamma,\rho} \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$ at (X) by Remark 9.3. Since $h^3(\mathcal{O}_X) = 1$ and $h^3(\mathcal{O}_X(-1)) = 24$, it follows that obsumext$(X) = 1 + \delta_X^4(-6) - 82 = -28$ by Theorem 9.1. Now we link twice to get X', first using a CI of type $(5,6)$, then a CI of type $(5,5)$, both times using a common hypersurface of degree 5. This is possible, cf. [12]. Thanks to (39) we find a free resolution of $I' = I_{X'}$ of the form

$$0 \to R(-7)^{\oplus 2} \to R(-6)^{\oplus 8} \to R(-5)^{\oplus 6} \oplus R(-4) \to I' \to 0. \tag{50}$$

By (28) $h^1(\mathcal{O}_{X'}(-2)) = 19$ and $h^1(\mathcal{O}_{X'}(-1)) = 0$ and we get $1 + \delta_{X'}^4(-6) = 20$. It follows from Theorem 9.1 and Proposition 6.1 that $\mathrm{H}_{\gamma',\rho'} \cong \mathrm{Hilb}^{p'(v)}(\mathbb{P}^5)$ is smooth at (X') of dimension

$$1 + \delta_{X'}^4(-6) - \mathrm{sumext}(X) = 48.$$

We can also use Proposition 9.9 and check that $1 - \delta_{X'}^0(0) = 48$.

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Minimal Links and a Result of Gaeta

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To the memory of Federico Gaeta

Abstract. If V is an equidimensional codimension c subscheme of an n-dimensional projective space, and V is linked to V' by a complete intersection X, then we say that V is *minimally linked* to V' if X is a codimension c complete intersection of smallest degree containing V. Gaeta showed that if V is any arithmetically Cohen-Macaulay (ACM) subscheme of codimension two then there is a finite sequence of minimal links beginning with V and arriving at a complete intersection. Gaeta's result leads to two natural questions.

First, in the codimension two, non-ACM case, there is no hope of linking V to a complete intersection. Nevertheless, an analogous question can be posed by replacing the target "complete intersection" with "minimal element of the even liaison class" and asking if the corresponding statement is true. Despite a (deceptively) suggestive recent result of Hartshorne, who generalized a theorem of Strano, we give a negative answer to this question with a class of counterexamples for codimension two subschemes of projective n-space.

On the other hand, we show that there *are* even liaison classes of non-ACM curves in projective 3-space for which every element admits a sequence of minimal links leading to a minimal element of the even liaison class. (In fact, in the classes in question, even and odd liaison coincide.)

The second natural question arising from Gaeta's theorem concerns higher codimension. In earlier work with Huneke and Ulrich, we showed that the statement of Gaeta's theorem as quoted above is false if "codimension two" is replaced by "codimension ≥ 3," at least for subschemes that admit a sequence of links to a complete intersection (i.e., *licci* subschemes). Here we show that in the non-ACM situation, the analogous statement is also false.

However, one can refine the question for codimension 3 licci subschemes by asking if it is true for *arithmetically Gorenstein, codimension 3* subschemes, which Watanabe showed to be licci. Watanabe's work was extended by Hartshorne, who showed that the *general* such subscheme of fixed Hilbert function and of dimension 1 can be obtained by a sequence of strictly ascending biliaisons from a linear complete intersection. (Hartshorne, Sabadini and

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Schlesinger proved the analogous result for arithmetically Gorenstein zero-dimensional schemes.) In contrast to the previous results, here we show that in fact for any codimension 3 arithmetically Gorenstein subscheme, in any projective space, a sequence of minimal links *does* lead to a complete intersection, giving a different extension of Watanabe's result. Furthermore, we extend Hartshorne's result by removing the generality assumption as well as the dimension assumption.

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1. Introduction

The study of liaison (or linkage) has a long history. The first important results were obtained by Gaeta, and subsequently the next flurry of activity occurred in the 70's and early 80's, and then in the last decade there has been possibly the biggest surge of interest. We refer to [25] and to [27] for a treatment of the basics of liaison theory, including an overview of what was known until roughly 2000. An important point, though, is that it is in many ways more natural to consider *even* liaison classes, i.e., the equivalent relation generated by linking an even number of times. (See below for details.) The theory has also branched out in the directions of Gorenstein liaison and complete intersection liaison. This paper deals exclusively with complete intersection liaison, and from now on unless otherwise specified, any link is assumed to be by a complete intersection.

In any even liaison class, a very important question has been to find distinguished elements of the class. This brings up two questions: what does "distinguished" mean, and how do you find such elements? We begin with the former question.

For certain subschemes of projective spaces, it is possible to arrive, after a finite number of links, to a complete intersection. These form the so-called *licci* class (for **li**nkage **c**lass of a **c**omplete **i**ntersection), and in this class we can view the complete intersections as being the most distinguished elements. An important question, still open in codimension ≥ 3, is to determine precisely which subschemes of projective space are licci. In codimension two, is was shown by Gaeta [11] that the licci subschemes are exactly the arithmetically Cohen-Macaulay (ACM) subschemes. It is his method of proof, described below, which inspired the work described in this paper.

For codimension two subschemes of projective space, it was shown in [22], [1], [6], [28], [29] in different settings that even liaison classes have a very precise structure, called the *Lazarsfeld-Rao property.* At the heart of this property is the notion of the *minimal* elements of the even liaison class. These subschemes simultaneously (!) are minimal with respect to degree, arithmetic genus, and shift of

the intermediate cohomology modules (see Definition 2.2). Furthermore, the entire even liaison class can be built up from the minimal elements in a prescribed way. There are actually two nearly equivalent ways to achieve this, but the one that we will mention here is that of *elementary biliaisons*. In our context, we will say that codimension two subschemes V and V' are related by an elementary biliaison if there is a hypersurface, S, containing both V and V', such that $V \sim V' + nH$ on S for some $n \in \mathbb{Z}$, where $\sim$ refers to linear equivalence and H is the class of a hyperplane section on S. The structure of the even liaison classes of curves in $\mathbb{P}^3$, using the elementary biliaison point of view, was achieved by Martin-Deschamps and Perrin [22]. It was extended by Hartshorne [15] to the case of codimension two in an arbitrary projective space. For curves in $\mathbb{P}^3$ it was extended further by Strano, who characterized the minimal curves in an even liaison class as being precisely those that do not admit an elementary descending biliaison class (i.e., V' is minimal above if n cannot be negative). This result was in turn extended again by Hartshorne (cf. Theorem 1.1 below).

In higher codimension, liaison theory forks giving *Gorenstein liaison* and *complete intersection* (CI) *liaison*. In the former case, it is no longer true that the notions of minimality mentioned above coincide – see [25] for a discussion. However, it is an open question for CI liaison. In any case, in this paper we will use the shift of the intermediate cohomology modules as our measure. See Section 2 for details.

Having described the set of distinguished elements of an even liaison class, we turn to the question of how one finds such elements. In the case of codimension two ACM subschemes of projective space, this was also solved by Gaeta in the process of showing that the ACM subschemes are licci. We now describe, in a more general setting, the construction that he (essentially) used for his result. Let V be a codimension c subscheme of projective space and let I_V be its homogeneous ideal. It is clear that there is a well-defined integer a_1 which is the least degree in which I_V is non-zero. More generally, there is a non-decreasing sequence of integers $a_1, a_2, \ldots, a_c$, where a_i is the least degree in which there is a regular sequence in I_V of length i. We say that a complete intersection X is a *complete intersection of least degree containing* V if $I_X \subset I_V$ and the generators of I_X have degrees $a_1, a_2, \ldots, a_c$. If V is linked by X to a subscheme V', we say that V is *minimally linked* to V'. Notice that this is not symmetric in general, since V' may admit a regular sequence with generators of smaller degree. Notice also that this extends in a natural way to the case of artinian ideals.

In the codimension two case, Gaeta proved that any ACM subscheme is minimally linked in a finite number of steps to a complete intersection, so in particular it is licci. In higher codimension it is known that not all ACM subschemes are licci. However, for those that are licci, it is natural to ask if there exists a sequence of minimal links arriving finally at a complete intersection. It was shown by Huneke, Ulrich and the current authors [20] that this is *not* the case: in any codimension ≥ 3, there exist subschemes which are licci, but cannot be minimally linked to a complete intersection. This paper builds on that work.

Gaeta also began the study of the non-ACM case by considering the curves $C \subset \mathbb{P}^3$ that have the property that a sequence of two minimal links can be found that returns to C (i.e., the first residual does not admit a regular sequence with generators of smaller degree). He referred to such a curve as being *ridotta seconda di se stessa*. The first author also studied such curves, using the term "locally minimal." He remarked ([23], page 130) that "it is natural to ask when locally minimal curves are actually minimal." The same question can very naturally be posed in codimension two in any projective space, and in a licci class in any codimension (but here instead of minimal elements, we simply ask whether the licci subscheme can be minimally linked to a complete intersection). It can also be posed in even greater generality, using the shift of the intermediate cohomology modules as the measure of minimality.

In view of the result for licci subschemes in [20] on the one hand, and Gaeta's result described above on the other hand, there are two natural questions that one can immediately pose, both of which are answered in this paper. First, what about the non-ACM case (both codimesion two and higher codimension)? Second, what about arithmetically Gorenstein subschemes? We now expand on these questions separately.

It is very natural to ask whether at least in codimension two, the process of linking by complete intersections of least degree can always be used to arrive at a minimal element in the even liaison class. This would be nice, for example, because it would give a natural and very elementary algorithm to obtain a minimal element in any such even liaison class. (More complicated algorithms do exist, e.g., [22], [13].) A very suggestive result in this direction is the following theorem of Hartshorne, which generalizes Strano's result mentioned above:

Theorem 1.1 ([15]**).** *If V is a codimension 2 subscheme whose degree is not minimal in its even liaison class, then V admits a strictly decreasing elementary biliaison.*

The amazing (to us) fact is that this does *not* imply that V can be minimally linked (in an even number of steps) to a minimal element. This is one of the two main areas of focus of this paper. First, we use a broad range of methods from liaison theory and minimal free resolutions to give a class of examples of codimension c subschemes in any $\mathbb{P}^n$ ($n \geq 3$, $c \geq 2$) that are not minimally linked in any number of steps to a minimal element of the even liaison class. More precisely, we have

Theorem 1.2. *Given any two integers $c \geq 2, d \geq 0$, there is a locally Cohen-Macaulay subscheme $X \subset \mathbb{P}^{c+d}$ of dimension d that is not a subscheme of any hyperplane of $\mathbb{P}^{c+d}$, such that X is not minimal in its even liaison class, but X cannot be linked to any minimal element in its even liaison class by way of* minimal *links. When $c = 2$, X is in fact linked in two steps (but not minimally linked) to a minimal element of its even liaison class.*

Furthermore, if $d \geq 0$ then X can be chosen as a licci subscheme, and if $d \geq 1$ it also can be chosen as a non-ACM subscheme.

If $c = 2$, then X must be necessarily non-ACM by Gaeta's theorem. The codimension two part of this result is given in Theorem 3.3. Since it was shown in [20] that in codimension ≥ 3 there are licci subschemes that are not *minimally* licci, it remains to produce non-ACM subschemes that cannot be minimally linked to a minimal element of their even liaison class. This is done in Proposition 4.1. The idea is to apply hypersurface sections of large degree to suitable subschemes with the same property (e.g., those obtained in Section 3), as was done in [20]. However, it is somewhat more delicate since the minimal generators do not behave as well under hypersurface sections as they do in the ACM case. The reason that we do not know that X is linked in two steps to a minimal element, in codimension ≥ 3, is that the hypersurface section of a minimal element is not necessarily minimal. Note also that taking cones over curves in $\mathbb{P}^3$ (where the constructions are simpler, as described in Theorem 3.6) is not sufficient to establish the above theorem in codimension two as the resulting subschemes are not locally Cohen-Macaulay.

Theorem 1.2 naturally leads to the question of whether the ACM class is the only liaison class of curves in $\mathbb{P}^3$ with the property that minimal links always lead to a minimal element. We show that this is not the case, by proving that the same property holds in the liaison class corresponding to a intermediate cohomology module that is n-dimensional (over the field k) and concentrated in one degree. The easiest example of such a liaison class is that of two skew lines (where $n = 1$).

We now turn to the arithmetically Gorenstein subschemes of codimension three. In many ways the known results about codimension three arithmetically Gorenstein subschemes closely parallel corresponding facts about codimension two ACM subschemes (see for instance [25], [9], [18], [12]). Furthermore, it is a result of Watanabe [34] that all arithmetically Gorenstein subschemes are licci. On the other hand, from [20] we know that codimension three licci subschemes are not necessarily minimally licci. So the codimension three arithmetically Gorenstein subschemes hang in the balance. In this paper we show that they are in fact minimally licci. The argument goes back to the structure theorem of Buchsbaum and Eisenbud, and a careful study of Watanabe's construction.

Watanabe's result has been extended for curves in $\mathbb{P}^4$ in [14] (resp. points in $\mathbb{P}^3$ in [17]) by showing that a *general* arithmetically Gorenstein curve (resp. set of points) with given Hilbert function can be obtained from a line (resp. single point) by a series of ascending complete intersection biliaisons. Since the result is obvious for complete intersections, the heart of the proof lies in showing that the general arithmetically Gorenstein curve in $\mathbb{P}^4$ (resp. set of points in $\mathbb{P}^3$) with given Hilbert function admits a series of strict descending complete intersection biliaisons down to a complete intersection. We also extend this result in Theorem 6.3 in two ways: we remove the generality condition, and we allow arbitrary $\mathbb{P}^n$. As noted above, results about strictly descending complete intersection biliaisons do not necessarily imply results about minimal links, but in this case both results do hold.

Gaeta in fact proved more for arithmetically Cohen-Macaulay codimension two subschemes. That is, if one makes links at each step which are not necessarily minimal but still use only minimal generators, one still obtains a complete

intersection in the same number of steps as occurs using minimal links. We show in Example 6.9 that this does not extend to the codimension three Gorenstein situation.

It is a satisfactory outcome that we were able to show that in some cases the minimal linkage property does hold, while in others it does not.

2. Some facts from liaison theory

Let $R = k[x_0, \dots, x_n]$, where k is a field. We refer to [25] for details of liaison theory, and here present just the basic facts that we will need.

Notation 2.1. If $\mathcal{F}$ is a sheaf on projective space $\mathbb{P}^n$, we denote by $H^i(\mathcal{F})$ the cohomology group $H^i(\mathbb{P}^n, \mathcal{F})$ and by $h^i(\mathcal{F})$ its dimension as a k-vector space. We denote by $H^i_*(\mathcal{F})$ the graded R-module $\bigoplus_{t\in\mathbb{Z}} H^i(\mathcal{F}(t))$.

Definition 2.2. If $V \subset \mathbb{P}^n$ is a closed subscheme, we define its *intermediate cohomology modules* (sometimes also called *Hartshorne-Rao modules* or *deficiency modules*) to be the graded modules $H^i_*(\mathcal{I}_V)$, for $1 \le i \le \dim V$.

If V_1 is an equidimensional subscheme of $\mathbb{P}^n$ and X is a complete intersection properly containing V_1 then X links V_1 to a *residual* subscheme $V_2 \subset \mathbb{P}^n$ defined by the saturated ideal $I_{V_2} = I_X : I_{V_2}$. The residual V_2 is also equidimensional, and because of the assumption that V_1 is equidimensional we have $I_X : I_{V_2} = I_{V_1}$ as well. The dimensions of V_1, V_2 and X all are equal. We write $V_1 \overset{X}{\sim} V_2$ and say that V_1 and V_2 are *directly linked by* X. Direct linkage generates an equivalence relation called *liaison*. If we further restrict to *even* numbers of links, we obtain the equivalence relation of *even liaison*.

Remark 2.3. In this paper we will always assume that our links are by complete intersections, as indicated above. However, there is a beautiful and active field of *Gorenstein liaison*, where X only needs to be arithmetically Gorenstein. This allows for many additional results that are not true when X must be a complete intersection, but it is also true that many interesting questions and results apply to this latter case, and this paper addresses some of these. When both possibilities are considered, it is sometimes useful to distinguish between *G-liaison* and *CI-liaison*. However, because of our restriction, we suppress the "CI" here.

It is well known that V is *arithmetically Cohen-Macaulay (ACM)* if and only if $H^i(\mathcal{I}_V) = 0$ for all $1 \le i \le \dim V$. More importantly, these modules are invariant up to shift in a non-ACM even liaison class:

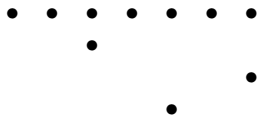
Theorem 2.4 (Hartshorne, Schenzel). *Let V_1 and V_2 be in the same* **even** *liaison class, and assume that they are not ACM. Then there is some integer δ such that*

$$H^i(\mathcal{I}_{V_1}) \cong H^i(\mathcal{I}_{V_2})(\delta) \quad \text{for } i = 1, \dots, \dim V_1 = \dim V_2.$$

We stress that the *same* δ is used for all values of i, so the intermediate cohomology modules move as a block under even liaison. It is not hard to see that there is a *leftmost* shift of this block that can occur in an even liaison class, while *any* rightward shift occurs [5]. We thus *partition* an even liaison class $\mathcal{L}$ according to the shift of the block of intermediate cohomology modules. Knowing the exact value of the leftmost shift is not needed – its existence is guaranteed, and the subschemes which achieve this leftmost shift are called the *minimal elements* of the even liaison class. In codimension two there is a structure common to all even liaison classes, called the *Lazarsfeld-Rao property* [22, 1, 28, 29], which is based on this partition. It says, basically, that the minimal elements are in fact also minimal with respect to the degree and the arithmetic genus, and furthermore that the entire even liaison class can be built up from any minimal element by a process called *basic double linkage*, together with flat deformations that preserve the block of cohomology modules.

At this point it is worth mentioning that, even for a licci subscheme of codimension two, a single minimal link does not necessarily produces a "smaller" residual subscheme.

Example 2.5. As an illustration, consider a set of points in $\mathbb{P}^2$ with the following configuration:



(There are 7 points on a line, and 3 sufficiently general points off the line.) Gaeta's theorem applies since Z is ACM of codimension two, so minimal links do lead to a complete intersection. However, our main measure of minimality, the degree, actually goes up with one minimal link. Specifically, the smallest link is with a regular sequence of type (3,7), so the residual set of points actually has degree 11. The next link, though, reduces to a single point.

The example illustrates the philosophy that pairs of links are more revealing than a single link and that one should study even liaison classes. Note however that in higher codimension the situation seems more complicated, even for Gorenstein liaison. In fact, a recent result of Hartshorne, Sabadini, and Schlesinger [17, Theorem 1.1] says that a general set of at least 56 points in $\mathbb{P}^3$ does not admit a strictly descending Gorenstein liaison or biliaison.

Since we will be using the construction of basic double linkage below, we briefly recall the important features for subschemes of codimension two. There are several versions and extensions of basic double linkage (for instance see [10, 21, 16]), but here we only require the version in codimension two, which is the simplest case. Let $V \subset \mathbb{P}^n$ be an equidimensional codimension 2 subscheme. Let $f \in I_V$ be a homogeneous polynomial and let ℓ be a linear form such that (ℓ, f) is a regular sequence. (It does not matter if ℓ vanishes on a component of V or not.) Then

the ideal $\ell \cdot I_V + (f)$ is the *saturated* ideal of a subscheme, Y, that is linked to V in two steps. In particular, the block of modules for Y is obtained from that of V by shifting one spot to the right. It is this mechanism that guarantees that all rightward shifts occur, as mentioned above. Geometrically, Y is the union of V with the (degenerate) complete intersection of ℓ and f.

Virtually all of the results in (complete intersection) liaison theory use, in one way or another, the so-called *Rao correspondence* between the even liaison classes and the stable equivalence classes of certain locally free sheaves [31, 32]. We will need certain aspects of this correspondence, which we now recall. Assume that $V \subset \mathbb{P}^n$ is a codimension two equidimensional *locally Cohen-Macaulay* subscheme. Then the ideal sheaf $\mathcal{I}_V$ has *locally free* resolutions of the form

$$0 \to \mathcal{F}_1 \to \mathcal{N} \to \mathcal{I}_V \to 0 \qquad (N\text{-type resolution})$$
$$0 \to \mathcal{E} \to \mathcal{F}_2 \to \mathcal{I}_V \to 0 \qquad (E\text{-type resolution})$$

where $H^1_*(\mathcal{E}) = 0$ and $H^{n-1}_*(\mathcal{N}) = 0$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ are direct sums of line bundles on $\mathbb{P}^n$. Thus, taking global sections we get short exact sequences of graded R-modules

$$0 \to F_1 \to N \to I_V \to 0 \qquad (N\text{-type resolution})$$
$$0 \to E \to F_2 \to I_V \to 0 \qquad (E\text{-type resolution})$$

where F_1 and F_2 are free R-modules.

We say that $\mathcal{E}_1$ and $\mathcal{E}_2$ (with the same hypotheses as $\mathcal{E}$) are *stably equivalent* if there exist direct sums of line bundles $\mathcal{F}_1, \mathcal{F}_2$ such that $\mathcal{E}_2 \oplus \mathcal{F}_2 \cong \mathcal{E}_1(c) \oplus \mathcal{F}_1$ for some integer c.

Theorem 2.6. [32] *Let $V_1, V_2 \subset \mathbb{P}^n$ be locally Cohen-Macaulay and equidimensional subschemes and let $\mathcal{N}_1$ (resp. $\mathcal{E}_1$) and $\mathcal{N}_2$ (resp. $\mathcal{E}_2$) be the locally free sheaves appearing in their N-type (resp. E-type) resolutions. Then V_1 and V_2 are in the same even liaison class if and only if $\mathcal{N}_1$ and $\mathcal{N}_2$ (resp. $\mathcal{E}_1$ and $\mathcal{E}_2$) are stably equivalent.*

This theorem has been extended to the non-locally Cohen-Macaulay case [28, 29, 15] but we do not need the extended version. A corollary of this theorem is that V (of codimension two in $\mathbb{P}^n$) is *licci* if and only if it is ACM. See the introduction for an explanation of licciness, as well as an improvement, due to Gaeta, involving minimal links.

Martin-Deschamps and Perrin [22] (for curves in $\mathbb{P}^3$) and later Hartshorne [15] (for codimension two subschemes of $\mathbb{P}^n$) gave a simplification of the Lazarsfeld-Rao property by replacing the deformation with constant cohomology by linear equivalence on a hypersurface. This is the notion of *biliaison*. Now our sequence of basic double links is replaced by a sequence of adding the class of a hyperplane (or hypersurface) section in the Picard group of the hypersurface. It is well known that if V is linearly equivalent on a hypersurface F to a divisor $V + dH$, where H is the class of a hyperplane, then V is linked in two steps to any element of the linear system $|V + dH|$. The difference between this and basic double linkage is that in

the latter case, we do not use linear equivalence, but rather use one deformation at the end of the sequence of adding to V hyperplane sections of (possibly a sequence of) hypersurfaces.

The Lazarsfeld-Rao property using biliaison allows the adding *and subtracting* of the class of a hypersurface section on hypersurfaces. However, the notion of *strictly ascending elementary biliaisons* refers to the fact that without loss of generality, we can restrict to *adding* hypersurface sections. In reverse, the notion in Theorem 1.1 of the existence of a *strictly decreasing elementary biliaison* refers to the assertion that if V is the divisor class of our subscheme on a hypersurface F then on F, $V - H$ is effective, where H is the class of a hyperplane. Theorem 1.1 asserts that whenever V is not minimal, there exists some hypersurface F on which a strictly decreasing elementary biliaison can take place.

3. Minimal linkage does not necessarily give minimal elements in codimension two

In this section we give a class of examples to show that starting with an arbitrary element of an even liaison class in codimension 2 and sequentially applying minimal links, one does not necessarily arrive at a minimal element of the even liaison class.

In order to produce such examples we use the correspondence of even liaison classes and certain reflexive modules. We begin by defining the modules we want to use. Let $R = k[x_0, \ldots, x_n]$, where $n \geq 3$, and consider the artinian module $M := R/(x_0^{n+1}, x_1, \ldots, x_n)$. Using its minimal free resolution (given by the Koszul complex), we define the modules N_1 and N by

$$\begin{array}{ccccccc} 0 \to R(-2n-1) \to F_n \to F_{n-1} & & \to & & F_{n-2} \to & \cdots \\ & \searrow & & \nearrow & & \\ & & N_1 & & & \\ & \nearrow & & \searrow & & \\ & 0 & & & 0 & \end{array}$$

$$\begin{array}{ccccc} \cdots \to F_2 & & \to & & F_1 \to R \to M \to 0, \\ & \searrow & & \nearrow & \\ & & N & & \\ & \nearrow & & \searrow & \\ & 0 & & & 0 \end{array}$$

where

$$F_{n-1} = R(-n+1)^n \oplus R(-2n+1)^{\binom{n}{2}} \quad \text{and} \quad F_2 = R(-2)^{\binom{n}{2}} \oplus R(-n-2)^n.$$

Next, we describe the minimal elements in the even liaison classes $\mathcal{L}_N$ and $\mathcal{L}_{N_1}$ corresponding to N and N_1, respectively.

Lemma 3.1.

(a) *Each minimal element, $I_{\min}$, in the even liaison class $\mathcal{L}_N$ has a minimal N-type resolution of the form*

$$0 \to R(-n+1)^{n-2} \oplus R(-2n+1) \to N(-n+3) \to I_{\min} \to 0 \tag{3.1}$$

and the shape of its minimal free resolution is

$$\begin{aligned} 0 \to R(-3n+2) \to F_n(-n+3) \to \cdots \to F_3(-n+3) \\ \to R(-n+1)^{\binom{n-1}{2}+1} \oplus R(-2n+1)^{n-1} \to I_{\min} \to 0. \end{aligned}$$

(b) *Each minimal element, $J_{\min}$, in the class $\mathcal{L}_{N_1}$ has a minimal N-type resolution of the form*

$$0 \to R(-n+1)^{n-2} \oplus R(-2n+1)^{\binom{n-1}{2}} \to N_1 \to J_{\min} \to 0 \tag{3.2}$$

and the shape of its minimal free resolution is

$$0 \to R(-2n-1) \to \begin{matrix} R(-n) \\ \oplus \\ R(-2n)^n \end{matrix} \to \begin{matrix} R(-n+1)^2 \\ \oplus \\ R(-2n+1)^{n-1} \end{matrix} \to J_{\min} \to 0.$$

Proof. (a) Since $I_{\min}$ is minimal in its even liaison class, its minimal N-type resolution has the form (see [22] or [28])

$$0 \to F \to N \to I_{\min}(c) \to 0,$$

where F is a free R-module such that the sum of the degrees of its minimal generators is minimal. The module N has rank n. Thus the module F is determined by a sequence of homogeneous elements $m_1, \ldots, m_{n-1} \in N$ of least degree that is $(n-1)$-fold basic in N at prime ideals of codimension at most one (see [28], Proposition 6.3). Notice that the module N has $\binom{n}{2}$ minimal generators of degree two and n generators of degree $n+2$. Moreover, the Koszul complex that resolves M shows that the generators of degree two generate a torsion-free R-module of rank $n-1$. Hence, we can find $n-2$ elements in N of degree two that form an $(n-2)$-fold basic sequence in N. Since besides the generators of degree two, N has only generators in degree $n+2$, this must be the degree of the $(n-1)$-st element in the $(n-1)$-fold basic sequence in N of least degrees. This means, the N-type resolution of $I_{\min}$ is of the form

$$0 \to R(-2)^{n-2} \oplus R(-n-2) \to N \to I_{\min}(c) \to 0.$$

The integer c is determined by the first Chern classes of F and N. Using the exact sequence

$$0 \to N \to F_1 \to R \to M \to 0$$

we get $c_1(N) = c_1(F_1) = -2n-1$. It follows that

$$\begin{aligned} c &= c_1(N) - c_1(F) \\ &= -2n-1-[-2(n-2)-n-2] = n-3. \end{aligned}$$

This establishes our assertion about the N-type resolution of $I_{\min}$. The claim about its minimal free resolution follows by using the mapping cone procedure, the minimal free resolution of N, and by observing that all the direct summands $R(-2)^{n-2} \oplus R(-n-2)$ of F will split because the corresponding generators map onto minimal generators of N.

(b) This is shown similarly. We only highlight the differences to the proof of part (a). This time $J_{\min}$ has a minimal N-type resolution of the form

$$0 \to G \to N_1 \to J_{\min}(c') \to 0,$$

where G is a free R-module of rank $\binom{n}{2} - 1$ with generators of least degrees. The minimal generators of N_1 have degrees $n-1$ and $2n-1$. Since the generators of degree $n-1$ generate a torsion-free module of rank $n-1$, it follows that $G \cong R(-n+1)^{n-2} \oplus R(-2n+1)^{\binom{n-1}{2}}$.

Using the defining sequence of N_1, a Chern class computation provides $c_1(N_1) = -\binom{n}{2}(2n-1) + (n+1)(n-1)$. Thus, we get

$$\begin{aligned} c' &= c_1(N_1) - c_1(G) \\ &= -\binom{n}{2}(2n-1) + (n+1)(n-1) - \left[-(n-2)(n-1) - \binom{n-1}{2}(2n-1)\right] = 0. \end{aligned}$$

This proves the assertion about the N-type resolution of $J_{\min}$. We get its minimal free resolution by using the mapping cone and observing the cancellation all of the direct summands $R(-n+1)^{n-2} \oplus R(-2n+1)^{\binom{n-1}{2}}$ of G. □

Corollary 3.2. *The even liaison classes $\mathcal{L}_N$ and $\mathcal{L}_{N_1}$ are residual classes.*

Proof. It suffices to show that $I_{\min}$ can be linked to a minimal element in $\mathcal{L}_{N_1}$. To this end we compare E-type resolutions.

The minimal free resolution of $I_{\min}$ shows that we can link it by a complete intersection of type $(n-1, 2n-1)$. Using the mapping cone procedure, we see that the residual J has a minimal E-type resolution

$$0 \to N^*(-2n-1) \to \begin{matrix} R(-n+1)^2 \\ \oplus \\ R(-2n+1)^{n-1} \end{matrix} \to J \to 0.$$

On the other hand, the self-duality of the resolution of M provides the short exact sequence

$$0 \to N^*(-2n-1) \to R(-n+1)^n \oplus R(-2n+1)^{\binom{n}{2}} \to N_1 \to 0.$$

Applying the mapping cone procedure to the N-type resolution of $J_{\min}$ and cancelling redundant terms, we get as its E-type resolution

$$0 \to N^*(-2n-1) \to \begin{matrix} R(-n+1)^2 \\ \oplus \\ R(-2n+1)^{n-1} \end{matrix} \to J_{\min} \to 0.$$

Comparing with the above E-type resolution of J, we see that J is a minimal element in $\mathcal{L}_{N_1}$, as claimed. □

We are ready for the main result of this section. Recall that a subscheme $X \subset \mathbb{P}^n$ is said to be *non-degenerate* if X is not a subscheme of any hyperplane of $\mathbb{P}^n$. Equivalently, this means that the homogeneous ideal I_X of X does not contain any linear form.

Theorem 3.3. *Let $f \in I_{\min}$ be a form of degree $n+1$ and let $\ell \in R$ be a linear form such that ℓ, f is an R-regular sequence. Let $X \subset \mathbb{P}^n$ be the codimension two subscheme defined by the saturated homogeneous ideal*

$$I_X = \ell \cdot I_{\min} + (f).$$

Then X is a non-degenerate, locally Cohen-Macaulay subscheme and I_X is in the even liaison class $\mathcal{L}_N$ of $I_{\min}$, but I_X cannot be linked to a minimal element in $\mathcal{L}_N$ by way of minimal links.

Proof. By definition I_X is a basic double link of $I_{\min}$. Thus, both ideals are in the same even liaison class. Moreover, the cohomology of X is determined by those of N. More precisely, one has

$$H^i_*(\mathbb{P}^n, \mathcal{I}_X) \cong \begin{cases} M & \text{if } i = 1 \\ 0 & \text{if } 2 \le i \le n-2 \end{cases}.$$

Hence X is locally Cohen-Macaulay.

Since I_X is defined by a basic double link, there is an exact sequence

$$0 \to R(-n-2) \to R(-n-1) \oplus I_{\min}(-1) \to I_X \to 0$$

(cf. [21]). Thus, the mapping cone procedure provides that I_X has a (not necessarily minimal) free resolution of the form

$$0 \to R(-3n+1) \to \cdots \to \begin{matrix} R(-n)^{\binom{n-1}{2}+1} \\ \oplus \\ R(-n-1) \\ \oplus \\ R(-2n)^{n-1} \end{matrix} \to I_X \to 0 \tag{3.3}$$

and an N-type resolution of the form

$$0 \to \begin{matrix} R(-n)^{n-2} \\ \oplus \\ R(-n-2) \\ \oplus \\ R(-2n) \end{matrix} \to \begin{matrix} R(-n-1) \\ \oplus \\ N(-n+2) \end{matrix} \to I_X \to 0. \tag{3.4}$$

Now, let I be any ideal which has a minimal free resolution and an N-type resolution as the ideal I_X above. We want to show:

Claim 1: I does not contain a regular sequence of type $(n, n+1)$.

Indeed, suppose this were not true. Then, we link I to an ideal J' by a complete intersection of type $(n, n+1)$. The residual J' has an E-type resolution

of the form (see, e.g., [28], Proposition 3.8)

$$0 \to \begin{matrix} R(-n) \\ \oplus \\ N^*(-n-3) \end{matrix} \to \begin{matrix} R(-1) \oplus R(-n+1) \\ \oplus \\ R(-n) \oplus R(-n-1)^{n-1} \end{matrix} \to J' \to 0.$$

It follows that the minimal free resolution of J' ends with a module generated in degree $n+3$, i.e., it is of the form

$$0 \to R(-n-3) \to \cdots \to J' \to 0.$$

However, thanks to Corollary 3.2, J' is in the even liaison class $\mathcal{L}_{N_1}$ and comparing with the minimal free resolution of $J_{\min}$, this gives a contradiction to the minimality of $J_{\min}$ because $n+3 < 2n+1$. Thus, Claim 1 is established.

From the free resolution of I we see that the degrees of the minimal generators of I are at most $n, n+1$, and $2n$, and that there is at least one minimal generator of degree n. Hence Claim 1 implies that the complete intersection of least degree inside I has type $(n, 2n)$. Let J be the residual of I with respect to this complete intersection. Its E-type resolution is of the form

$$0 \to \begin{matrix} R(-2n+1) \\ \oplus \\ N^*(-2n-2) \end{matrix} \to \begin{matrix} R(-n)^2 \\ \oplus \\ R(-2n+2) \\ \oplus \\ R(-2n)^{n-1} \end{matrix} \to J \to 0. \tag{3.5}$$

We want to find the minimal link of J.

Claim 2: J does not contain a complete intersection of type $(n, 2n-2)$.

Indeed, otherwise we could link J by a complete intersection of type $(n, 2n-2)$ to an ideal I' with N-type resolution

$$0 \to \begin{matrix} R(-n+2)^{n-1} \\ \oplus \\ R(-n) \\ \oplus \\ R(-2n+2)^2 \end{matrix} \to \begin{matrix} R(-n+1) \oplus R(-n) \\ \oplus \\ R(-2n+2) \\ \oplus \\ N(-n+4) \end{matrix} \to I' \to 0.$$

Comparing the degree shift of N in the N-type resolution of I' with those in the resolution of $I_{\min}$, we get a contradiction to the minimality of $I_{\min}$ because $n-4 < n-3$. This establishes Claim 2.

The free resolution (3.5) shows that the ideal J is generated in degrees $n, 2n-2$, and $2n$. Since J has at most one generator of degree $2n-2$, the claim implies that the minimal link of J is given by a complete intersection of type $(n, 2n)$. Denote the residual by $\tilde{I}$. Using the cancellation due to the fact that the minimal generators of the complete intersection are also minimal generators of J, we see

that $\tilde{I}$ has an N-type resolution of the form

$$0 \to \begin{array}{c} R(-n)^{n-2} \\ \oplus \\ R(-n-2) \\ \oplus \\ R(-2n) \end{array} \to \begin{array}{c} R(-n-1) \\ \oplus \\ N(-n+2) \end{array} \to \tilde{I} \to 0.$$

Since the generators of $R(-n)^{n-2}$ map onto minimal generators of $N(-n+2)$, the mapping cone procedure provides (after taking into account the resulting cancellation) that $\tilde{I}$ has a free resolution of the form

$$0 \to R(-3n+1) \to \cdots \to \begin{array}{c} R(-n)^{\binom{n-1}{2}+1} \\ \oplus \\ R(-n-1) \\ \oplus \\ R(-2n)^{n-1} \end{array} \to \tilde{I} \to 0.$$

Comparing this with resolutions (3.3) and (3.4), we see that any ideal I with N-type and minimal free resolution as I_X is minimally linked in two steps to an ideal with the same resolutions. Hence I (and also J) cannot be minimally linked to a minimal element in its even liaison class. □

Remark 3.4. (i) The arguments in the above theorem can easily be modified to provide other examples of subschemes that cannot be minimally linked to a minimal element in its even liaison class. Indeed, instead of starting with the module $M := R/(x_0^{n+1}, x_1, \ldots, x_n)$, one could use $R/(x_0^d, x_1, \ldots, x_n)$ for some $d \geq n+1$. Defining the modules N_1 and N analogously, one can then take the basic double link of a minimal element in the class $\mathcal{L}_N$ on a hypersurface of suitable degree to get further examples with a the same behavior as X. However, we restricted ourselves to the specific choices above in order to keep the arguments as simple as possible.

(ii) Further examples can be obtained by taking cones. Notice however, that contrary to the examples we constructed, these cones are no longer locally Cohen-Macaulay.

In the case of curves in $\mathbb{P}^3$, simpler constructions and arguments suffice to make our conclusions. Omitting the proofs (which are simplifications of the ones given above), the following can be shown.

Proposition 3.5. *Let C_1 be a curve in $\mathbb{P}^3$ consisting of the disjoint union of a line and a plane curve, Y, of degree d, with $d \geq 1$. Then*

(a) *$M(C_1) \cong k[w]/(g)$, where $g(w)$ is a polynomial of degree d. In particular,*

$$\dim M(C)_t = \begin{cases} 1 & \text{if } 0 \leq t \leq d-1; \\ 0 & \text{otherwise.} \end{cases}$$

(b) *C_1 is minimal in its even liaison class.*

(c) *The minimal free resolution of* I_{C_1} *is of the form*

$$0 \to R(-d-3) \xrightarrow{\sigma} \begin{matrix} R(-3) \\ \oplus \\ R(-d-2)^3 \end{matrix} \to \begin{matrix} R(-d-1)^2 \\ \oplus \\ R(-2)^2 \end{matrix} \to I_{C_1} \to 0.$$

(d) $M(C_1)$ *is self-dual, hence the even liaison coincides with the entire liaison class. In particular, two curves that are directly linked are also evenly linked.*

(e) *The minimal link for* C_1 *is a complete intersection of type* $(2, d+1)$*. Such a complete intersection links* C_1 *to another minimal curve in the (even) liaison class.*

Theorem 3.6. *Choose integers* d, e *satisfying*

$$4 \leq e \leq d.$$

Let C_1 *be the disjoint union of a line and a plane curve,* Y*, of degree* d*. Let* f *be a homogeneous element of* I_{C_1} *of degree* e*. Observe that* f *necessarily contains as a factor the linear form of* I_Y*. Let* ℓ *be a general linear form, and define* C *by the saturated homogeneous ideal*

$$I_C = \ell \cdot I_{C_1} + (f).$$

The curve C *cannot be linked to a minimal curve in its even liaison class by way of minimal links.*

Remark 3.7. Since minimal curves in any even liaison class are minimally linked to minimal curves in the residual liaison class (cf. [22], Theorem IV.5.10), it follows from Theorem 3.6 that C is not minimally linked to a minimal curve in the residual class either. (In any case, this module is self-dual so the minimal curves coincide.)

Remark 3.8. In [19], Huneke and Ulrich showed that if an ideal I in a local Gorenstein ring with infinite residue field is licci, then one can pick regular sequences consisting of general linear combinations of minimal generators at every linkage step to reach a complete intersection. Gaeta's theorem may be viewed as an analogous result for projective ACM subschemes of codimension two. Theorem 3.3 shows that, in general, its conclusion is not longer true if one drops the assumption that the schemes are ACM.

Remark 3.9. It is not true that if a curve is irreducible then it is minimally linked to a minimal curve. Indeed, the curve C produced in Theorem 3.6 can be linked using two generally chosen surfaces of degree 12 to a residual curve, C', which is smooth and for which the smallest complete intersection is again of type (12,12). Hence C' is minimally linked to C, and we have seen that C is not minimally linked to a minimal curve. However, it is an open question whether we can find an irreducible curve C that is minimally linked in two steps back to a curve that is numerically the same as C in the sense of Theorem 3.6.

Remark 3.10. We now make the connection between Hartshorne's Theorem 1.1 and our results above, and in particular why the latter do not contradict the former. For simplicity we will use the context of curves in $\mathbb{P}^3$, and Theorem 3.6,

for our discussion, but it holds equally well in the codimension two setting. It is clear (as Hartshorne's theorem guarantees) that C admits a strictly descending elementary biliaison. It does so on the surface F defined by f, and one obtains the minimal curve C_1 as a result: $C - H = C_1$ on F, where H is the class of a hyperplane section of F. But this only guarantees a pair of links of the form (f, a) and (f, b), where a and b are forms and $\deg b < \deg a$. One can show that C does not admit a link of type $(3, d)$. Using the assumption $4 \leq e \leq d$, it follows that neither f nor a can have degree 3. Hence Hartshorne's theorem says nothing about the minimal link of C in this case, since a minimal link necessarily involves a surface of degree 3.

What about the possibility that in addition to the elementary biliaison mentioned above there is another one that involves a minimal link as the first step? Recall that a minimal link for C is of type $(3, d+2)$. Suppose that there is a strictly descending elementary biliaison on a surface G, with $\deg G = d + 2$ or $\deg G = 3$, where the first link is minimal. That means, in either case, that after performing the first link (as was done in the proof) one obtains a residual that allows a smaller link. This is precisely what was proved to be impossible in Theorem 3.6.

4. Hypersurface sections

In this section we show that in any $\mathbb{P}^n$ $(n \geq 4)$ there exist non-ACM, locally Cohen-Macaulay subschemes of any codimension c with $2 \leq c \leq n - 1$ for which no sequence of minimal links reaches a minimal element of the even liaison class. The last section showed this result when $c = 2$, so this will be our starting point. Our method of attack will be via hypersurface sections of large degree, in a manner analogous to that used in [20] for the licci class, although here some new ideas are needed. Since non-ACM schemes have at least one non-zero intermediate cohomology module, our measure of minimality will be by the shift of the collection of intermediate cohomology modules (see Section 2).

If X is at least a surface, we will show that under suitable conditions we can take hypersurface sections and preserve the linkage property. So let $X \subset \mathbb{P}^n$ be a non-ACM, codimension $c - 1$ subscheme with $\dim X \geq 2$ and satisfying the following properties:

(i) X is not minimal in its even liaison class.
(ii) X has the property that no sequence of minimal links arrives at a minimal element of the even liaison class (and hence, as noted in the last section, no sequence of minimal links arrives at a minimal element of the residual even liaison class either).
(iii) $H^1_*(\mathbb{P}^n, \mathcal{I}_X) \cong (R/I)(-\delta)$ for some artinian ideal I and some $\delta > 0$. Let e be the socle degree of R/I, i.e., the last degree in which R/I is non-zero.
(iv) $H^i_*(\mathbb{P}^n, \mathcal{I}_X) = 0$ for all $2 \leq i \leq \dim X$.

Note that X is automatically locally Cohen-Macaulay and equidimensional, thanks to the conditions on the cohomology of $\mathcal{I}_X$.

Proposition 4.1. *Let X be as above. Fix an integer $d > e$ (see* (iii)*) and assume also that $d+\delta$ is greater than any degree of a minimal generator of I_X. Let F be a general form of degree d. Let Y be the hypersurface section of X cut out by F, so I_Y is the saturation of (I_X, F). Then Y also satisfies the property that no sequence of minimal links starting with Y arrives at a minimal element of the liaison class.*

Proof. We have taken F to be a general form of degree d. The meaning of "general" will be made more precise as we go through the proof, but at each step it will be clear that each new constraint is still an open condition, and there are finitely many of them. The first condition, of course, is that F meets each component of X properly.

Note first that since $H^1_*(\mathbb{P}^n, \mathcal{I}_X) \neq 0$, R/I_X has depth 1, and the last free module in the minimal free resolution of R/I_X is the same as that in the minimal free resolution of $H^1_*(\mathbb{P}^n, \mathcal{I}_X)$. The twist of this last free module measures the shift of the intermediate cohomology module, and hence measures how far X is from being minimal in its even liaison class.

We first claim that I_Y has two more generators than I_X does: one is F, and the other comes from the intermediate cohomology module. As an intermediate step, we have to consider the ideal of Y in the hypersurface defined by F, i.e., $I_{Y|F}$. We use some ideas from [25].

Consider the exact sequence of sheaves,

$$0 \to \mathcal{I}_X(-d) \xrightarrow{\times F} \mathcal{I}_X \to \mathcal{I}_{Y|F} \to 0. \tag{4.1}$$

Since $d \geq e$, multiplication by F is zero on R/I. Hence we have the following long exact sequence in cohomology (taking direct sums over all twists):

$$\begin{array}{ccccccccc} 0 & \to & I_X(-d) & \xrightarrow{\times F} & I_X & & \longrightarrow & I_{Y|F} & \to (R/I)(-\delta - d) \to 0 \\ & & & & & \searrow & & \nearrow & \\ & & & & & & \frac{I_X}{F \cdot I_X} & & \\ & & & & & \nearrow & & \searrow & \\ & & & & 0 & & & 0 & \end{array} \tag{4.2}$$

because F annihilates R/I by the assumption about its degree. Since $\frac{I_X}{F \cdot I_X}$ has the same degrees of minimal generators as I_X and R/I has one minimal generator, the Horseshoe Lemma applied to the last short exact sequence coming from (4.2) shows that $I_{Y|X}$ has exactly one extra generator (besides those coming from the restriction of I_X to F), coming in degree $d+\delta$. The embedding $(F) \hookrightarrow I_Y$ induces the short exact sequence

$$0 \to R(-d) \xrightarrow{\times F} I_Y \to I_{Y|F} \to 0,$$

which shows that I_Y has one additional minimal generator, namely F itself. To conclude:

One minimal generator of I_Y is F, and one has degree $d+\delta$. The remaining degrees of minimal generators of I_Y coincide with those of I_X, and in any minimal generating set for $I_{Y|F}$, the generators of $I_{Y|F}$ corresponding to these remaining degrees all lift to minimal generators of I_X (by degree considerations).

Note that the above observations apply to any X' in the same even liaison class, and with the same degrees of generators, as X.

Recall that F was chosen to be a "general" form of degree d; we now add a requirement for this generality. We have assumed that X is not minimal in its even liaison class, which means that X can be linked in some sequence of (non-minimal) links down to one which has a more leftward shift of the deficiency modules. We assume that F meets all of these intermediate links, including the linking hypersurfaces, properly. It follows that Y is not minimal in its even liaison class: simply adjoin F to each of the complete intersections used in the sequence of links for X, to obtain a subscheme which has a more leftward shift of the deficiency modules than Y. Here we have used again the fact that F annihilates all the intermediate cohomology modules of the subschemes participating in the links.

Now suppose that $(G_1, \dots, G_c)$ is a minimal complete intersection in I_Y. In particular, we may take the G_i to be part of a minimal generating set for I_Y. Since $I_X + (F)$ is generated in degrees $< d+\delta$, all of these G_i have degrees $< d+\delta$. Furthermore, without loss of generality we may take one of them, say G_c, to be F. As for the rest, if we consider their restriction to $I_{Y|F}$, they necessarily lift to I_X. Let Y' be defined by $I_{Y'} = (G_1, \dots, G_c) : I_Y$. Say $(G_1, \dots, G_{c-1}) \subset I_Y$ restricts to $(\bar{G}_1, \dots, \bar{G}_{c-1}) \subset I_{Y|F}$, which in turn lifts to $(F_1, \dots, F_{c-1}) \subset I_X$. Clearly this is a minimal link for I_X. Consider the residual subscheme X' defined by $I_{X'} = (F_1, \dots, F_{c-1}) : I_X$. By the Hartshorne-Schenzel theorem, $H^i_*(\mathcal{I}_{X'}) = 0$ for $1 \le i \le n-3$. In particular, $R/I_{X'}$ has depth ≥ 2. Hence $(I_{X'}, F)$ is saturated, and $I_{Y'} = (I_{X'}, F)$. Then any minimal generating set of $I_{Y'}$ can (without loss of generality) be written as F together with the restrictions of minimal generators of $I_{X'}$.

It follows that a minimal link for $I_{Y'}$ lifts to a minimal link for X'. By the numerical conditions that we have set, any finite sequence of minimal links starting with Y will lift to a sequence of minimal links starting with X. Since the latter never allow for a smaller link, the same is true for the former. □

Combining with the main result of Section 3 we obtain

Corollary 4.2. *In any codimension there are locally Cohen-Macaulay, equidimensional, non-ACM subschemes that are not minimal in their even liaison class, and cannot be minimally linked in a finite number of steps to a minimal element.*

Proof. We simply observe that from the exact sequence (4.1) it also follows that Y satisfies properties (iii) and (iv) before Proposition 4.1. Properties (i) and (ii) come directly from Proposition 4.1, so we may successively apply that proposition as

many times as we like as long as the scheme that we are cutting with a hypersurface is at least of dimension 2. To start the process we have the main result of Section 3. □

Of course this is not surprising, given the main result of [20], but still an example needed to be found.

5. A non-arithmetically Cohen-Macaulay extension of Gaeta's theorem

In view of Gaeta's result on the one hand and Theorem 3.6 on the other, it is natural to ask if the arithmetically Cohen-Macaulay case is the only one where every curve is minimally linked to a minimal curve. In this section we show that this is not the case!

For the following theorem we will consider the liaison class of curves in $\mathbb{P}^3$ whose corresponding intermediate cohomology module is isomorphic to k^n, concentrated in one degree. We will call this liaison class $\mathcal{L}_n$. Note that since the module k^n is self-dual, any two curves in $\mathcal{L}_n$ are both evenly and oddly linked. The following provides some facts that we will need.

Lemma 5.1. *Let $C \in \mathcal{L}_n$. The following are equivalent.*

1. *C is a minimal curve in $\mathcal{L}_n$.*
2. *$\deg C = 2n^2$.*
3. *$M(C)$ is non-zero in degree $2n-2$.*
4. *I_C has a minimal free resolution*

$$0 \to R(-2n-2)^n \to R(-2n-1)^{4n} \to R(-2n)^{3n+1} \to I_C \to 0.$$

Proof. Parts (1), (2) and (3) are from [3], also applying the Lazarsfeld-Rao property [22], [1], while (4) is an easy calculation using [31]. □

Theorem 5.2. *Let C be a curve in the liaison class $\mathcal{L}_n$. Then C is minimally linked to a minimal curve (in a finite number of steps).*

Proof. The graded module k^n has minimal free resolution

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R(-4)^n & \xrightarrow{\sigma} & R(-3)^{4n} & \longrightarrow & R(-2)^{6n} & \longrightarrow & R(-1)^{4n} & \longrightarrow & \\ & & & & & \searrow & & \nearrow & & \searrow & \\ & & & & & & E & & E^*(-4) & & \\ & & & & & \nearrow & & \searrow & & \nearrow & \\ & & & & 0 & & & 0 & & 0 & \\ & & & & & & & & & R^n & \longrightarrow k^n \longrightarrow 0. \end{array} \tag{5.1}$$

Let a be the initial degree of I_C and let b be the first degree in which I_C has a regular sequence of length two. Thanks to a theorem of Rao [31], I_C has a minimal

free resolution of the form

$$\begin{array}{ccccccccc} & & R(-e+1)^{4n} & & & & R(-a)\oplus R(-b) & & \\ 0 \to R(-e)^n \to & & \oplus & & \longrightarrow & & \oplus & \longrightarrow I_C \to 0 \\ & & \mathbb{F} & \searrow & & \nearrow & \mathbb{G} & & \\ & & & & E(4-e)\oplus\mathbb{F} & & & & \\ & & 0 & \nearrow & & \searrow & 0 & & \end{array} \tag{5.2}$$

where $\mathbb{F}$ and $\mathbb{G}$ are free modules, and σ is the homomorphism coming from the minimal free resolution of k^n. Note that the intermediate cohomology module, $M(C)$, of C occurs in degree $e-4$.

The strategy of the proof is as follows. Note that as usual, if C is linked by a complete intersection of type (a,b) then the smallest complete intersection containing the residual, C', is at most of type (a,b). If the assertion of the theorem is false, then we would eventually come to a curve C for which the smallest complete intersection containing the residual, C', is again of type (a,b). *We will show that if this situation arises then C is already a minimal curve, and $a=b=2n$.* To do this, we look for numerical conditions that guarantee that C' allows a smaller link, and rule out these numerical conditions. This is restrictive enough that it forces C to be minimal. There are two such kinds of numerical conditions that we will use: (i) all the minimal generators of $I_{C'}$ occur in degree $< b$, and (ii) the initial degree of $I_{C'}$ is $< a$.

We first find a free resolution of the residual curve, $I_{C'}$. Using the E-type resolution of I_C indicated in (5.2) and splitting the summands corresponding to $R(-a)$ and $R(-b)$, we see that $I_{C'}$ has an N-type resolution of the form

$$0 \to \mathbb{G}^*(-a-b) \to \begin{array}{c} E^*(e-4-a-b) \\ \oplus \\ \mathbb{F}^*(-a-b) \end{array} \to I_{C'} \to 0.$$

From (5.1) we know a minimal free resolution of E^*. Hence an application of the mapping cone gives a free resolution (not necessarily minimal) of $I_{C'}$:

$$\begin{array}{l} 0 \longrightarrow R(-4+e-a-b)^n \xrightarrow{(\sigma,0)} \begin{array}{c} R(-3+e-a-b)^{4n} \\ \oplus \\ \mathbb{G}^*(-a-b) \end{array} \longrightarrow \\ \qquad\qquad\qquad\qquad \begin{array}{c} R(-2+e-a-b)^{6n} \\ \oplus \\ \mathbb{F}^*(-a-b) \end{array} \longrightarrow I_{C'} \longrightarrow 0. \end{array} \tag{5.3}$$

Now, assume that $\mathbb{F} \neq 0$ and let $\mathbb{F} = \bigoplus_{i=1}^r R(-c_i)$. Set

$$c = \max\{c_i\}, \qquad d = \min\{c_i\}.$$

Since the smallest twist of the free modules in the minimal free resolution of I_C (5.2) is strictly increasing, we obtain

$$e \geq a+2, \qquad d \geq a+1 \tag{5.4}$$

Since $E^*(e-4-a-b)$ is not a summand of $\mathbb{G}^*(-a-b)$, not all summands of $R(-2+e-a-b)^{6n}$ split off in (5.3). In particular,

$$I_{C'} \text{ has at least one generator of degree } a+b+2-e.$$

Also, no summand of $\mathbb{F}^*(-a-b)$ splits off. Still assuming $\mathbb{F} \neq 0$, it follows that the degrees of the generators of $I_{C'}$ corresponding to $\mathbb{F}^*(-a-b)$ range from $a+b-c$ to $a+b-d$. In particular, using (5.4), we have that the generators corresponding to $\mathbb{F}^*(-a-b)$ (if there are any) all have degree $\leq b-1$.

Suppose that $e > a+2$. Then $a+b+2-e < b$. Combining with the previous paragraph, we see that $I_{C'}$ has no minimal generator of degree $\geq b$, and this is also true if $\mathbb{F}$ is trivial. It follows that C' allows a smaller link. Thus combining with (5.4), we obtain that

$$\text{without loss of generality we may assume } e = a+2.$$

Now we again consider the minimal free resolution (5.2), as well as the free resolution of $I_{C'}$, using this new substitution:

$$0 \to R(-a-2)^n \to \begin{matrix} R(-a-1)^{4n} \\ \oplus \\ \mathbb{F} \end{matrix} \to \begin{matrix} R(-a) \\ \oplus \\ R(-b) \\ \oplus \\ \mathbb{G} \end{matrix} \to I_C \to 0 \tag{5.5}$$

and

$$0 \to R(-b-2)^n \to \begin{matrix} R(-b-1)^{4n} \\ \oplus \\ \mathbb{G}^*(-a-b) \end{matrix} \to \begin{matrix} R(-b)^{6n} \\ \oplus \\ \mathbb{F}^*(-a-b) \end{matrix} \to I_{C'} \to 0. \tag{5.6}$$

Recall that $\mathbb{F} = \bigoplus_{i=1}^r R(-c_i)$ (or 0) and $c = \max\{c_i\}$. Suppose that $\mathbb{F} \neq 0$ and suppose that $b < c$. We noted above that no summand of $\mathbb{F}^*(-a-b)$ splits off. Using that $c \geq d \geq a+1$, we see that the smallest generator of $I_{C'}$ has degree $a+b-c < a$, so again C' allows a smaller link. We thus have two possibilities:

1. $\mathbb{F} = 0$, in which case $I_{C'}$ has $\leq 6n$ generators, all of degree b. Since $a \leq b$ and $I_{C'}$ contains elements of degree a, it follows that $a = b$.
2. $\mathbb{F} \neq 0$, in which case we need $b \geq c$ by the discussion in the preceding paragraph. But then considering the minimal free resolution (5.5) and using that the generators of $\mathbb{F}$ have degrees at most $c \leq b$, in order for the generator(s) of I_C having degree b to participate in any syzygy, we again need $a = b$ so that the syzygies of degree $a+1$ can apply to the generator(s) of degree b. But then any generator of $\mathbb{F}$ has degree $\leq a$ and any generator of $\mathbb{G} \oplus R(-a)^2$ has degree $\leq a$. This is a contradiction to the minimality of the resolution.

We are left with the conclusion that $\mathbb{F} = 0$ and $a = b$. Hence I_C has minimal free resolution

$$0 \to R(-a-2)^n \to R(-a-1)^{4n} \to R(-a)^{3n+1} \to I_C \to 0$$

(where the $3n+1$ comes from considering the ranks of the free modules). But then considering the twists, we obtain $n(a+2) + (3n+1)a = 4n(a+1)$, from which it follows that $a = 2n$. This is the minimal free resolution of the minimal curve, and so C is minimal as claimed, thanks to Lemma 5.1. □

Remark 5.3. We believe that other liaison classes of curves in $\mathbb{P}^3$ possess the property that every curve is minimally linked (in a finite number of steps) to a minimal curve, but it seems that numerical considerations of this sort will not be enough. Indeed, we considered two natural next cases. Both have a two-dimensional module, with one-dimensional components in each of two consecutive degrees. Specifically, we first considered

- Y_1 is a minimal Buchsbaum curve with this module (meaning that the dimensions are as given, but the structure as a graded module is trivial). Then thanks to [4], we know that $\deg Y_1 = 10$ and $M(Y_1)$ occurs in degrees 2 and 3. This curve is easy to construct with liaison addition [33].
- Y_2 is the disjoint union of a line and a conic (which is not Buchsbaum). This curve is minimal in its even liaison class (see Proposition 3.5). The liaison properties of such curves were studied in [24].

Note that in both cases, the intermediate cohomology module is self-dual up to shift. In both of these cases, almost the entire argument given above was able to go through. However, using the resulting constraints as guidelines, in the end we were able to use basic double linkage starting from the minimal curve (in a careful way) to provide a *non-minimal* curve C in the corresponding liaison class, which is minimally (directly) linked to a curve C', and such that C and C' have the following properties:

- C and C' are cohomologically indistinguishable: they have the same degree and arithmetic genus, the same Hilbert function, and their intermediate cohomology modules occur in the same degrees.
- I_C and $I_{C'}$ have minimal free resolutions that are almost indistinguishable: the generators occur in degrees a and $a+1$, but I_C possesses an extra copy of $R(-a-1)$ in the first two free modules in the resolution (a so-called ghost term).
- C is minimally linked to C' by a complete intersection of type $(a, a+1)$, but nevertheless, C' allows a complete intersection of type (a, a), linking it to a minimal curve.

Remark 5.4. It should not be hard, using our methods, to find a collection of codimension two even liaison classes in any projective space such that every element is minimally linked to a minimal element.

We end this section with a natural question:

Question 5.5. Which liaison classes $\mathcal{L}$ of curves in $\mathbb{P}^3$ have the property that every curve in $\mathcal{L}$ is minimally linked (in a finite number of steps) to a minimal element of $\mathcal{L}$? We note that "most of the time" there are actually two families of minimal curves, corresponding to the two even liaison classes in $\mathcal{L}$, but it was shown by Martin-Deschamps and Perrin [22] that any minimal curve is minimally linked to a minimal curve in the residual class.

6. Gorenstein ideals of height three

A by now classical theorem of Watanabe says that every Gorenstein ideal of height three is licci. This is shown by induction on the number of minimal generators using the following result:

Lemma 6.1 ([34])**.** *Let $I \subset R$ be a homogeneous Gorenstein ideal of height three that is not a complete intersection. Let $f, g, h \in I$ be a regular sequence such that f, g, h can be extended to a minimal generating set of I. Then I is linked by the complete intersection (f, g, h) of height three to an almost complete intersection $J = (f, g, h, u)$.*

Assume furthermore that u, f, g is a regular sequence. Then J is linked by the complete intersection (u, f, g) to a Gorenstein ideal I' whose number of minimal generators is two less than the number of minimal generators of I.

Proof. This result is not stated in [34]. However, it is shown in the proof of [34, Theorem] if I is a Gorenstein ideal in a regular local ring R. The same arguments work for a homogeneous ideal I in a polynomial ring R over a field. □

Note that the passage from I to I' in this statement is an elementary biliaison. It is a strictly descending biliaison if $\deg u < \deg h$.

For curves, Hartshorne complemented Watanabe's result by showing the following (see also [17] for points in $\mathbb{P}^3$):

Theorem 6.2 ([14])**.** *Every* general *Gorenstein curve in $\mathbb{P}^4$ can be obtained from a complete intersection by a sequence of strictly ascending elementary biliaisons.*

The goal of this section is to strengthen both, Watanabe's and Hartshorne's result.

Theorem 6.3. *Let $I \subset R$ be a homogeneous Gorenstein ideal of height three. Then:*

(a) *If I is not a complete intersection then linking I minimally twice gives a Gorenstein ideal with two fewer generators than I.*
(b) *Consequently, I is minimally licci.*
(c) *Furthermore, I admits a sequence of strictly decreasing CI-biliaisons down to a complete intersection.*

The key to this result is to identify a particular choice for the element u in Watanabe's lemma. Not surprisingly, its proof relies on the structure theorem of

Buchsbaum and Eisenbud [7]. The main result in [7] says that every Gorenstein ideal of height three is generated by the submaximal Pfaffians of an alternating map between free modules of odd rank. This means that each such Gorenstein ideal I corresponds to a skew-symmetric matrix M whose entries on the main diagonal are all zero and whose number of rows is odd such that the ith minimal generator of I is the Pfaffian of the matrix obtained from M by deleting row and column i of M. We refer to M as the *Buchsbaum-Eisenbud matrix* of I.

Lemma 6.4. *Let $I \subset R$ be a homogeneous Gorenstein ideal of height three with Buchsbaum-Eisenbud matrix M. Assume that I is not a complete intersection, and let $f, g, h \in I$ be a regular sequence such that f, g, h are the Pfaffians of the matrix obtained from M by deleting row and column i, j, and k, respectively. Then*

$$(f, g, h) : I = (f, g, h, u),$$

where u is the Pfaffian of the matrix obtained from M by deleting rows and columns i, j, k.

Proof. This is essentially a consequence of [7, Theorem 5.3]. For the convenience of the reader we provide some details.

Let $s \geq 5$ be the number of minimal generators of I. Then, by [7], I has a minimal free resolution of the form

$$0 \to \bigwedge^{s} F \to \bigwedge^{s-1} F \to F \to I \to 0,$$

where F is a free R-module of rank s and the middle map corresponds to an element $\varphi \in \bigwedge^2 F$. It provides the exterior multiplication $\lambda : \bigwedge F \to \bigwedge F,\ a \mapsto a \wedge \varphi^{(\frac{s-3}{2})}$, where $\varphi^{(j)}$ denotes the jth divided power of φ.

Consider the minimal free resolution of the complete intersection (f, g, h):

$$0 \to \bigwedge^{3} G \to \bigwedge^{2} G \to G \to (f, g, h) \to 0.$$

The inclusion $\iota : (f, g, h) \hookrightarrow I$ induces a map $\alpha : G \to F$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigwedge^3 G & \longrightarrow & \bigwedge^2 G & \longrightarrow & G & \longrightarrow & (f,g,h) & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \bigwedge^s F & \longrightarrow & \bigwedge^{s-1} F & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0.
\end{array}$$

Buchsbaum and Eisenbud have determined the remaining comparison maps (see [7, page 474]) such that one gets a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigwedge^3 G & \longrightarrow & \bigwedge^2 G & \longrightarrow & G & \longrightarrow & (f,g,h) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_3 \circ \bigwedge^3 \alpha} & & \downarrow{\scriptstyle \lambda_2 \circ \bigwedge^2 \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \bigwedge^s F & \longrightarrow & \bigwedge^{s-1} F & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0.
\end{array}$$

Now observe that the map $\lambda_3 \circ \bigwedge^3 \alpha$ is the multiplication by an element, say, $u \in R$. Hence a mapping cone argument (see [30, Proposition 2.6]) provides that

$$(f, g, h) : I = (f, g, h, u),$$

It remains to identify the element u. By our assumption $\{f, g, h\}$ can be extended to a minimal generating set. Hence, using suitable bases the map $\alpha : G \to F$ can be described by a matrix whose columns are part of the standard basis of F. Therefore the map $\bigwedge^3 \alpha : \bigwedge G \to \bigwedge^3 F$ is given by a matrix with one column whose entries are all zero except for one, which is 1. The map $\lambda_3 : \bigwedge^3 F \to \bigwedge^s F$ is given by a matrix with one row whose entries are the Pfaffians of order $s-3$ of M. It follows that the product of these two matrices is the form $u \in R$ that equals the Pfaffian of the matrix obtained from M by deleting each of the rows and columns that lead to $(s-1) \times (s-1)$ matrices whose Pfaffians are f, g, and h, respectively. This completes the argument. □

We are ready for the proof of the main result of this section.

Proof of Theorem 6.3. We may assume that I is not a complete intersection. Using induction on the number of minimal generators we will prove all statements simultaneously by showing that two consecutive minimal links constitute a strictly descending elementary CI-biliaison.

Let (f, g, h) be a complete intersection of height three and of least degree inside I. Then $\{f, g, h\}$ can be extended to a minimal generating set of I. Thus, we may apply Lemma 6.4 and we will consider the corresponding element u. Order the degrees such that $\deg f \leq \deg g \leq \deg h$. Note that the assumption that I is not a complete intersection forces $\deg f \geq 2$. We need another estimate:

Claim 1: $\deg u < \deg g$.

Indeed, we may order the Buchsbaum-Eisenbud matrix M of I such that the degrees of its entries are increasing from bottom to top and from right to left. Then the entries on the non-main diagonal must all have positive degree. Since (f, g, h) is a complete intersection of least degree inside I, the degree of f is the least degree of a minimal generator of I. Thus, we may assume, using the notation of Lemma 6.4, that $i = 1 < j < k$. Developing the Pfaffian that gives g along its left-most column, Lemma 6.4 yields that $\deg g$ is the sum of $\deg u$ and the degree of the $(k, 1)$ entry of M, which is positive. It follows that $\deg g > \deg u$, as claimed.

By the choice of $f, g, h \in I$, the ideal I is minimally linked to $J = (f, g, h, u)$. The next goal is to show:

Claim 2: u, f is an R-regular sequence.

Suppose otherwise. Then there are forms $a, b, c \in R$ such that $u = ab,\ f = ac$, and $\deg a \geq 1$. By symmetry of linkage, the complete intersection $(f, g, h) = (ac, g, h)$ links $J = (ab, ac, g, h)$ back to I, but one easily checks that $(ac, g, h) : (ab, ac, g, h)$ contains c, contradicting our assumption that the least degree of an element in I is $\deg f$. This completes the proof of Claim 2.

Now let $\mathfrak{c}$ be a height 3 complete intersection of least degree inside J. The two claims above imply that $\mathfrak{c}$ is of the form (u, f', g'), where $\deg f' = \deg f$ and $\deg g'$ equals $\deg g$ or $\deg h$. Moreover, we may assume that $f', g' \in (f, g, h)$. In order to complete the argument we need:

Claim 3: $\{f', g'\}$ can be extended to a minimal generating set of (f, g, h).

To this end we distinguish two cases:

Case 1: Assume $\deg g' = \deg g$.
Then choose $h' \in (f, g, h)$ sufficiently general of degree $\deg h$. Since (f, g, h) has height 3, (f', g', h') will also be a complete intersection of height 3. Moreover, it is contained in (f, g, h), and both intersections have the same degree. It follows that $(f', g', h') = (f, g, h)$, as desired.

Case 2: Assume $\deg g' = \deg h > \deg g$.
Then we may assume that there are forms $a, b \in R$ such that $g' = af + bg + h$. Now we choose $h' \in (f, g)$ sufficiently general of degree $\deg g$. Then (f', h') will be a complete intersection of height 2 inside (f, g). Both intersections have the same degree, thus we get $(f', h') = (f, g)$. It follows that $(f', g', h') = (f, af + bg + h, h')$ equals (f, g, h), as wanted.

Thus, Claim 3 is established.

By the choice of $\mathfrak{c}$, the ideal J is minimally linked by $\mathfrak{c} = (u, f', g')$ to an ideal I'. Since I is minimally linked by (f, g, h) to J, Claim 3 combined with Watanabe's Lemma 6.1 shows that I' is a Gorenstein ideal with two fewer minimal generators. As pointed out above, Claim 3 also shows that the passage from I to I' may be viewed as an elementary CI-biliaison. It is strictly decreasing because $\deg f' = \deg f$, $\deg g \leq \deg g' \leq \deg h$, and $\deg u < \deg g$, thus

$$\deg u + \deg f' + \deg g' < \deg f + \deg g + \deg h.$$

This completes the proof. □

Remark 6.5. (i) The above result is not stated in its utmost generality at all. In fact, Theorem 6.3 is true whenever I is a homogeneous Gorenstein ideal of height three in a graded Gorenstein algebra R over any field K, where the assumption includes that the projective dimension of I over R is finite.

(ii) Parts (a) and (b) of Theorem 6.3 are also true for every Gorenstein ideal of height three in a local Gorenstein ring R.

(iii) If we were only interested in parts (b) and (c), Lemma 6.4 (or even Lemma 6.1) would not be necessary, and the main ingredients of a much shorter proof are already contained in the above proof. However, in proving (a) we have the additional benefit of getting at the heart of the structure of Gorenstein ideals of codimension three with respect to liaison.

(iv) There are many instances in the literature illustrating the principle that codimension two arithmetically Cohen-Macauay ideals and codimension three arithmetically Gorenstein ideals have properties that closely parallel one another. (See [25] Section 4.3 for an extensive list with references.) Theorem 6.3 is another

illustration – part (a) is also true for codimension two Cohen-Macaulay ideals, and indeed it is Gaeta's approach to proving his theorem.

We illustrate our result by giving various examples.

Example 6.6. Let Z be a set of 8 points on a twisted cubic curve in $\mathbb{P}^3$. Then Z is arithmetically Gorenstein, with h-vector $(1,3,3,1)$. The smallest link for Z is of type $(2,2,3)$, so the residual, Z', has h-vector $(1,2,1)$. One can check that the smallest link for Z' is of type $(1,2,3)$ (as a codimension three subscheme of $\mathbb{P}^3$), so we have that $1 = \deg u < \deg f = 2$.

Example 6.7. Let Z be an arithmetically Gorenstein subscheme of $\mathbb{P}^3$ constructed as follows. Choose a point P in $\mathbb{P}^3$ and let Λ be a plane through P and let λ be a line through P not in Λ. Let X be a complete intersection of type $(3,3)$ in Λ containing P, and let $Z_1 = X \backslash P$. Let Z_2 be a set of four points on λ, none equal to P. Then $Z = Z_1 \cup Z_2$ is arithmetically Gorenstein [2] with minimal free resolution

$$0 \to R(-7) \to \begin{matrix} R(-3) \\ \oplus \\ R(-4)^2 \\ \oplus \\ R(-5)^2 \end{matrix} \to \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3)^2 \\ \oplus \\ R(-4) \end{matrix} \to I_Z \to 0$$

and h-vector $(1,3,4,3,1)$. One easily checks that Z has a minimal link of type $(2,3,4)$, linking Z to an almost complete intersection Z' with minimal free resolution

$$0 \to \begin{matrix} R(-6) \\ \oplus \\ R(-7) \end{matrix} \to \begin{matrix} R(-4)^2 \\ \oplus \\ R(-5)^2 \\ \oplus \\ R(-6) \end{matrix} \to \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \\ \oplus \\ R(-4) \end{matrix} \to I_{Z'} \to 0.$$

We first note that Z' also has h-vector $(1,3,4,3,1)$, rather than having smaller degree than Z. However, by Claim 2 above, Z' has a regular sequence of type $(2,2)$. We saw that Z' is not a complete intersection. Since Z' has degree 12, the minimal link for Z' must be of type $(2,2,4)$. One checks that Z' is then linked to a complete intersection of type $(1,2,2)$. In this example we have $\deg u = 2 = \deg f$.

Example 6.8. In Example 6.7, suppose X is a complete intersection of type $(6,6)$ on Λ and Z_2 is a set of 10 points on λ, then $Z = Z_1 \cup Z_2$ is again arithmetically Gorenstein, this time with minimal free resolution

$$0 \to R(-13) \to \begin{matrix} R(-3) \\ \oplus \\ R(-7)^2 \\ \oplus \\ R(-11)^2 \end{matrix} \to \begin{matrix} R(-2)^2 \\ \oplus \\ R(-6)^2 \\ \oplus \\ R(-10) \end{matrix} \to I_Z \to 0.$$

A minimal link is of type $(2, 6, 10)$, with residual Z' having minimal free resolution

$$0 \to \begin{matrix} R(-12) \\ \oplus \\ R(-16) \end{matrix} \to \begin{matrix} R(-7)^2 \\ \oplus \\ R(-11)^2 \\ \oplus \\ R(-15) \end{matrix} \to \begin{matrix} R(-2) \\ \oplus \\ R(-5) \\ \oplus \\ R(-6) \\ \oplus \\ R(-10) \end{matrix} \to I_{Z'} \to 0.$$

In this case $\deg u = 5 > 2 = \deg f$.

Example 6.9. For codimension two arithmetically Cohen-Macaulay subschemes, Gaeta in fact proved the stronger result that if one always links using minimal generators, not necessarily minimal links, it still holds that the ideal obtained in the second step has two fewer minimal generators than the original ideal (in fact, one link provides an ideal with one fewer minimal generator, but this does not have an analog here). Hence even weakening the minimal link condition to simply links by minimal generators, we can still conclude that the ideal is licci. We now show that this does not extend to the Gorenstein situation. Let I be a Gorenstein ideal of height 3 with a minimal free resolution of the form:

$$0 \to R(-7) \to R(-4)^7 \to R(-3)^7 \to I \to 0.$$

Linking I minimally by a complete intersection of type $(3, 3, 3)$, we get an ideal J with minimal free resolution:

$$0 \to R(-6)^4 \to R(-5)^7 \to \begin{matrix} R(_3)^3 \\ \oplus \\ R(-2) \end{matrix} \to J \to 0.$$

Now we choose three sufficiently general cubic forms in J. They generate a complete intersection $\mathfrak{c}$ and are minimal generators of J. Linking J by $\mathfrak{c}$ we get a Gorenstein ideal I' whose resolution has the same shape as the one of I. In particular, I and I' have the same number of minimal generators. Notice that the second link to get I' is not minimal, thus this example does not contradict Theorem 6.3(a). However, this example shows that to draw the conclusion of Theorem 6.3(a), it is not enough to assume only that the two links both use minimal generators of the "starting" ideal.

Remark 6.10. As mentioned in the introduction, Theorem 6.3 generalizes a result of Hartshorne [14] and a result of Hartshorne, Sabadini and Schlesinger [17], in addition to sharpening Watanabe's result.

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On the Existence of Maximal Rank Curves with Prescribed Hartshorne-Rao Module

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Abstract. In this paper, we study maximal rank curves $C \subset \mathbb{P}^3$ in connection to their cohomology and natural lifting properties. More precisely, we set $R = K[x, y, z, t]$ and we address the following 4 problems:

(1) To characterize the graded finite length R-modules M such that there is a maximal rank curve in the biliaison class of M.
(2) To characterize the graded finite length R-modules M such that there is a maximal rank curve C in the biliaison class of M with the lifting property $s(C) = \sigma(C)$.
(3) To characterize the graded finite length R-modules M such that there is a maximal rank curve C in the biliaison class of M with the lifting property $s(C) = \sigma^*(C)$.
(4) To characterize the Rao functions of the maximal rank curves C, with the lifting property $s(C) = \sigma(C)$ (resp. $s(C) = \sigma^*(C)$).

We partially answer the problems (1)–(3). Problem (4) seems to be more difficult and we can only describe some classes of Rao functions associated to some maximal rank curves.

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1. Introduction

The aim of this paper is to study maximal rank space curves in connection with their cohomology and with two natural lifting properties.

We work in the projective space $\mathbb{P}^3 = \mathrm{Proj}(R)$, where $R = K[x, y, z, t]$, K an algebraically closed field of characteristic zero. Moreover by curve we mean a pure one-dimensional (hence locally Cohen-Macaulay) closed subscheme $C \subseteq \mathbb{P}^3$.

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We are interested in the following problems (see Section 2 for precise definitions and references):

Problem 1.1. To characterize the graded finite length R-modules M such that there is a maximal rank curve in the biliaison class of M.

Problem 1.2. To characterize the graded finite length R-modules M such that there is a maximal rank curve C in the biliaison class of M with the further lifting property $s(C) = \sigma(C)$, where s(C) is the least degree of a surface containing C and $\sigma(C)$ is the least degree of a curve containing a general plane section of C.

Problem 1.3. Same as problem 1.2 with $\sigma(C)$ replaced by $\sigma^*(C)$, the least degree of a curve containing any plane sections of C (and not only the general ones).

Problem 1.4. To characterize the sequences $h^1(\mathcal{I}_C(j))$ (the so-called Rao functions) of the maximal rank curves C, with or without the lifting properties of Problems 1.2 and 1.3.

Our first contribution to these problems is a sufficient conditions for Problem 1.1, namely:

Theorem 1.5. *Let M be a non-zero graded R-module of finite length. Let b be the least degree of a second syzygy of M and let q be the largest degree of a non-zero element of M. If $b \geq q+1$ then there is a maximal rank curve C in the biliaison class of M.*

This follows from the more detailed Theorem 3.5 proved in Section 3, where we show also that the converse of Theorem 1.5 is false in general (Example 3.6).

Our answers to Problems 1.2 and 1.3 are summarized in the following Theorem, which follows from Theorem 3.9.

Theorem 1.6. *Let M, b and q be as in Theorem* 1.5 *and assume that $b \geq q+1$. Then in the biliaison class of M there is a maximal rank curve X satisfying $s(X) = \sigma(X)$ (resp. $s(X) = \sigma^*(X)$) if and only if the linear map $M_{q-1} \to M_q$ induced by a general (resp. every) linear form is injective.*

Concerning Problem 1.4 we can only describe some classes of Rao functions, mostly with the assumption that M is cyclic.

We begin with Koszul modules (see Proposition 4.2). We see that only three Rao functions are allowed, namely, up to shift: $1, 0, \ldots$; $1, 1, 0, \ldots$ and $1, 2, 1, 0, \ldots$, and that only the first two allow the lifting property $s(C) = \sigma(C)$.

Next we deal with some classes of Rao functions, related to more general cyclic modules. For example we get:

Theorem 1.7. *For a fixed integer $k > 0$ there are maximal rank k-Buchsbaum curves* (see §2 for definition) *C satisfying $s(C) = \sigma(C)$ and having any of the following Rao functions (up to shift):*

(i) $1, 4, \ldots, \binom{k+2}{3}, 0, \ldots$ *(the largest compatible with a cyclic module);*
(ii) $1, 4, \ldots, \binom{k+2}{3} - \epsilon, 0, \ldots$ *with $0 < \epsilon \leq \binom{k+1}{2}$ (the next largest);*
(iii) $1, 3, \ldots, \binom{k+1}{2} - \epsilon, 0, \ldots$ *with $0 \leq \epsilon < \binom{k+1}{2}$ (one of the smallest).*

This follows from Proposition 4.3 and Corollaries 4.6 and 4.10.

Then we deal with Rao functions of the form $1, 4, \ldots, 1+3(k-1), 0, \ldots$, which are quite relevant in connection with Problem 1.3. Here we can settle the question only for $k \leq 4$, see Propositions 4.11 and 4.12. For $k = 4$ we need non-cyclic modules.

The paper ends with two results where non-cyclic modules are used. These concern the characterization of the Rao functions that occur for maximal rank 1-Buchsbaum curves with the lifting property $s = \sigma^*$ (Proposition 4.14) and a relevant property concerning the diameter of M for maximal rank curves (Example 4.15).

2. Preliminaries

We establish some notation and recall some basic facts. First of all K, R and $\mathbb{P}^3$ are as in the introduction, and our basic reference is [5].

For a graded R-module M we set $p(M) := \min\{j \in \mathbb{Z} \cup \{\pm\infty\} \mid M_j \neq 0\}$ and $q(M) := \max\{j \in \mathbb{Z} \cup \{\pm\infty\} \mid M_j \neq 0\}$ (so $p(M) = q(M) = -\infty$ iff $M = 0$).

For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$, we set $s(\mathcal{F}) := p(H^0_*(\mathcal{F}))$.

Given a closed subscheme $X \subseteq \mathbb{P}^n$, we denote by $\mathcal{I}_X$ the ideal sheaf of X and we set $s(X) := s(\mathcal{I}_X)$; we often write s in place of $s(X)$ if X is clearly understood. If H is a hyperplane we put $s(X \cap H) := s(\mathcal{I}_{X\cap H,H})$ and we define: $\sigma(X) := s(H \cap X)$, where H is a general hyperplane and $\sigma^*(X) := \min\{s(H \cap X)\}$ as H varies in the set of hyperplanes not containing any irreducible component of X_{red}. Clearly we have $s(X) \geq \sigma(X) \geq \sigma^*(X)$, the latter by semicontinuity. We write simply σ and σ^* if no confusion can arise.

By a curve we mean a closed equidimensional, locally Cohen-Macaulay, one-dimensional subscheme of $\mathbb{P}^3$.

Let $C \subseteq \mathbb{P}^3$ be a curve. The *Hartshorne-Rao module* (or *deficiency module*) of C $M_C := \bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t))$ is a graded R-module of finite length; we denote by ρ_C the *Rao function* of C, i.e., $\rho_C(t) := \dim_K(M_C)_t = h^1(\mathcal{I}_C(t))$ and following [9] we set $r_a = r_a(C) := p(M_C)$ and $r_o = r_o(C) := q(M_C)$ (then $r_a = r_o = -\infty$ iff C is arithmetically Cohen-Macaulay).

A curve $C \subset \mathbb{P}^3$ has *maximal rank* if the restriction map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(t)) \to H^0(C, \mathcal{O}_C(t))$ has maximal rank, i.e., $h^0\mathcal{I}_C(t) \cdot h^1\mathcal{I}_C(t) = 0$ for all $t \in \mathbb{Z}$. This is equivalent to the inequality $s(C) > r_o(C)$.

Remark 2.1. Let C be a non arithmetically Cohen-Macaulay curve and set $s := s(C)$, $\sigma := \sigma(C)$, $\sigma^* := \sigma^*(C)$. Then by using the cohomology sequence

$$0 \longrightarrow H^0(C, \mathcal{I}_C(j)) \longrightarrow H^0(C, \mathcal{I}_C(j+1)) \longrightarrow H^0(C \cap H, \mathcal{I}_{C\cap H}(j+1)) \longrightarrow H^1(C, \mathcal{I}_C(j)) \longrightarrow H^1(C, \mathcal{I}_C(j+1))$$

associated to the hyperplane sequence

$$0 \longrightarrow \mathcal{I}_C(-1) \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_{C\cap H} \longrightarrow 0 \tag{1}$$

it is easy to show the following:

(i) *The linear map* $(M_C)_j \to (M_C)_{j+1}$ *induced by a general linear form is injective for* $j \leq \sigma - 2$*;*
(ii) $\rho_C(j) \leq \rho_C(j+1)$ *for* $j \leq \sigma - 2$*;*
(iii) $r_o \geq \sigma - 1$*;*
(iv) $s = \sigma$ *(resp.* $s = \sigma^*$*) if and only if the linear map* $(M_C)_{s-2} \to (M_C)_{s-1}$ *induced by a general linear form (resp. by any linear form not vanishing on any irreducible component of* C_{red}*) is injective;*
(v) *if* C *has maximal rank and* $s = \sigma$ *then* $r_o = s - 1$*. Moreover* (i) *and* (ii) *hold for all* $j < r_o(C)$ *(this follows by* (iii)*).*

The behavior of the Rao function for values $j < \sigma^*$ has more restrictions. Indeed we have:

Lemma 2.2. *Let the notation be as in Remark* 2.1*. Then we have:*

(i) ρ_C *is strictly increasing in the interval* $[r_a, \sigma^* - 1]$*;*
(ii) *if* C *contains no lines (resp. no planar subcurves), then the Rao function is increasing by at least* 2 *(resp.* 3*), in the interval* $[r_a, \sigma^* - 1]$*, i.e., for all* $r_a \leq j < \sigma^* - 1$ *we have* $\rho_C(j) + 2 \leq \rho_C(j+1)$ *(resp.* $\rho_C(j) + 3 \leq \rho_C(j+1)$*);*
(iii) *If* C *has maximal rank and* $s(C) = \sigma^*(C)$ *the above hold for all* $r_a \leq j < r_o$*.*

Proof. Let $E := \{\ell \in R_1 \mid \ell \text{ does not vanish on any component of } C_{\mathrm{red}}\}$. It is easy to see that E contains a linear space of V with $\dim V \geq 2$ and moreover with $\dim V \geq 3$ (resp. $\dim V = 4$) if C contains no lines (resp. no planar subcurves). The bilinear map $\phi : V \times (M_C)_t \to (M_C)_{t+1}$ induced by multiplication is non-degenerate for $r_a \leq t < \sigma^* - 1$ (i.e., for $r_a \leq t < \sigma^* - 1$, for each $0 \neq v_1 \in V$ and each $0 \neq v_2 \in (M_C)_t$, $\phi(v_1, v_2) \neq 0$) . Then by the bilinear Lemma (see [6], Lemma 5.1) we have $\dim(M_C)_{t+1} \geq \dim(M_C)_t + \dim V - 1$, whence (i) and (ii). The last statement follows by Remark 2.1(iii). □

Let M be a graded R-module of finite length.The *Buchsbaum index* of M is $k(M) := \min\{t \mid \mathfrak{m}^t M = 0\}$. If $k = k(M)$ we also say that M is k-Buchsbaum. Thus M is 0-Buchsbaum if and only if it is the zero module.

If M is non-zero the *diameter* of M, written $\operatorname{diam} M$, is the number of components of M from the first non-zero one to the last (inclusive); hence $\operatorname{diam} M = q(M) - p(M) + 1$. Note that $k(M) \leq \operatorname{diam} M$.

A curve C is said to be k-Buchsbaum if $k = k(M_C)$. For example C is 0-Buchsbaum (resp. 1-Buchsbaum) if and only if C is arithmetically Cohen-Macaulay (resp. arithmetically Buchsbaum).

Let C be k-Buchsbaum with $k > 0$. If C has maximal rank there are heavy restrictions for $\operatorname{diam} M_C$. Indeed we have:

Proposition 2.3. *Let* C *be a* k*-Buchsbaum curve,* $k \geq 1$*, of maximal rank. Then:*

(i) ([11], Proposition 2.5) $k \leq \operatorname{diam}(M_C) \leq k + 1$*;*
(ii) *If* $s(C) = \sigma(C)$ *then* $k = \operatorname{diam}(M_C)$*.*

Proof. We have to prove only (ii). By Remark 2.1 the map $H^1(\mathcal{I}_C(s-1-k)) \to H^1(\mathcal{I}_C(s-1))$ induced by the k^{th} power of a general linear form is injective. Since this map is also the zero map by definition of k we have that $H^1(\mathcal{I}_C(s-1-k)) = 0$, whence $k \geq \text{diam}(M_C)$. Now since C has maximal rank the conclusion follows from (i). □

We will use freely the basic notions of liaison theory, see [10] or [9] for reference. If M is a finite length graded R-module we denote by $\mathbb{B}(M)$ the biliaison class determined by M, namely the set of all curves whose Hartshorne-Rao module is isomorphic to M, up to shift. It is well known that $\mathbb{B}(M) \neq \emptyset$. More precisely $\mathbb{B}(M)$ is the union of a countable set of flat families; for each shift of M we get a a countable set of flat families of curves unless for a certain leftmost one, which corresponds to the so called minimal curves in $\mathbb{B}(M)$ which form a unique flat family (see, for instance, [2]; Corollary 5.7).

3. Proofs of the main results

In this section we prove (a more complete version of) Theorems 1.5 and 1.6 of the introduction, along with some Corollaries.

We begin with some notation and with two preliminary Lemmas.

Notation 3.1. Throughout this section M will be a non-zero graded R-module of finite length, with $q := q(M)$ and $0 = p(M)$ (note that this is not restrictive). Moreover:

(i) We denote the beginning of a minimal free resolution of M as follows:

$$\cdots \to \bigoplus_{i=1}^{r} R(-b_i) \to \bigoplus_{i=1}^{n} R(-a_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{m} R(-p_i) \to M \to 0 \tag{2}$$

where $0 = p_1 \leq p_2 \leq \cdots \leq p_m$, $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_r$.

(ii) We set $b = b(M) := b_1$, the least degree of a second syzygy of M.

(iii) Recall that the Castelnuovo-Mumford regularity of M is exactly q (see, e.g., [4], Corollary 4.4). Hence $a_n \leq q+1$ and $b_r \leq q+2$. In particular $b \leq q+2$.

(iv) We set $\mathcal{E} := ker\tilde{\varphi}$. It is a locally free sheaf fitting into the exact sequence

$$0 \to \mathcal{E} \to \bigoplus_{i=1}^{n} \mathcal{O}(-a_i) \xrightarrow{\tilde{\varphi}} \bigoplus_{i=1}^{m} \mathcal{O}(-p_i) \to 0 \tag{3}$$

deduced from (2).

Lemma 3.2. *Let $\mathcal{F}$ be a locally free sheaf of rank $r > 1$ over $\mathbb{P}^3$, generated by global sections. Assume further that $\mathcal{F}$ has no direct summand isomorphic to $\mathcal{O}$. Then there is a non-singular curve $X \subseteq \mathbb{P}^3$ whose ideal sheaf fits into an exact sequence*

$$0 \to \mathcal{O}^{r-1} \to \mathcal{F} \to \mathcal{I}_X(c_1(\mathcal{F})) \to 0 \tag{4}$$

where the first morphism is given by the choice of $r-1$ general elements of $H^0(\mathcal{F})$.

Proof. Let $\psi : \mathcal{O}^{r-1} \to \mathcal{F}$ be a general morphism and let X be its degeneracy scheme. If $X = \emptyset$ then $\mathcal{L} := \operatorname{coker}\psi$ is locally free of rank 1. Then by duality $\operatorname{Ext}^1(\mathcal{L}, \mathcal{O}^{r-1}) = 0$ whence $\mathcal{F}$ has a direct summand isomorphic to $\mathcal{O}$, a contradiction. Thus $X \neq \emptyset$. Then $\operatorname{codim} X \leq 2$ by general facts on determinantal ideals, whence X is a smooth curve by [3], Main Theorem. The conclusion follows using the Eagon-Northcott complex. □

Lemma 3.3. *Let the notation be as in* 3.1. *Then we have:*

(i) $H^1_*(\mathcal{E}) = M$, $H^2_*(\mathcal{E}) = 0$ *and* $H^3(\mathcal{E}(j)) = 0$ *for* $j > q - 3$;
(ii) $\mathcal{E}(q+2)$ *is generated by global sections;*
(iii) $s(\mathcal{E}) = b \leq q + 2$ *and* $\operatorname{rk}\mathcal{E} \geq 2$;
(iv) *There is a smooth curve* C *whose ideal sheaf fits into an exact sequence*

$$0 \to \mathcal{O}^{n-m-1} \to \mathcal{E}(q+2) \to \mathcal{I}_C(t) \to 0 \tag{5}$$

where the first morphism is determined by a general choice of $n - m - 1$ *sections of* $\mathcal{E}(q+2)$ *and* $t = c_1(\mathcal{E}(q+2))$;
(v) $M_C = M(-t+q+2)$, $r_o(C) = t - 2$ *and* $r_a(C) = t - 2 - q$;
(vi) $s(C) = t - (q + 2 - b) \leq t$;
(vii) C *has maximal rank if and only if* $b \geq q+1$, *if and only if* $q+1 \leq b \leq q+2$. *Moreover if* $b = q + 2$ *then* C *is minimal in* $\mathbb{B}(M)$;
(viii) $\sigma(C) \leq t - 1$;
(ix) $t \geq q + 2$.

Proof. (i) From(3) we have immediately the first two equalities. Moreover by 3.1(iii) we have $a_i \leq q + 1$ for every i, whence the third by duality.

(ii) By (i) we have $\operatorname{reg}(\mathcal{E}) \leq q+2$, whence the conclusion (see [12], lecture 14 or [4], Corollary 4.8).

(iii) From (3) we have the exact sequence of graded R-modules:

$$0 \to H^0_*(\mathcal{E}) \to \bigoplus_{i=1}^{n} R(-a_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{m} R(-p_i) \to M \to 0$$

which implies $s(\mathcal{E}) = b$. Moreover $b \leq q + 2$ by 3.1(iii). The same sequence easily implies the last assertion, since M has homological dimension 4.

(iv) $\mathcal{E}$ has no direct summand of rank 1 by [9], p. 40, Proposition 2.4. The conclusion follows then by Lemma 3.2.

(v) It follow from (i) and the exact sequence (5).

(vi) By (iii) we have $q + 2 - b \geq 0$, whence the inequality. Now from (5) twisted by $-(q + 2 - b)$ we get:

$$h^0(\mathcal{I}_C(t - (q + 2 - b))) = h^0(\mathcal{E}(b)) - h^0(\mathcal{O}^{n-m-1}(-(q + 2 - b))). \tag{6}$$

Now (6) and (iii) imply $h^0(\mathcal{I}_C(t-(q+2-b))) > 0$. This is immediate if $q+2-b > 0$. On the other hand if $q + 2 - b = 0$ it is sufficient to observe that by (ii) we have $h^0(\mathcal{E}(b)) = h^0(\mathcal{E}(q+2)) \geq \operatorname{rk}\mathcal{E} > n - m - 1 = h^0(\mathcal{O}^{n-m-1}(-(q + 2 - b)))$.

The same argument (twisting by $-(q+2-b)-1$) shows that $h^0(\mathcal{I}_C(t-(q+2-b)-1)) = 0$, whence the conclusion.

(vii) The first assertion follows immediately from (vi) and (5). Moreover it is easy to see that by (5) the speciality index $e(C) := q(H^1_*(\mathcal{O}_C))$ is $e(C) = t - 4$. Hence if $b = q + 2$ by (vi) we have $s(C) = t > e(C) + 3$, whence C is minimal by [9], p. 89, Proposition 4.7 (see also [8], Theorem 1.4).

(viii) Follows from (v) and Remark 2.1(iii).

(ix) Since C is smooth it is reduced. Hence $r_a(C) \geq 0$ and the conclusion follows from (v). □

Remark 3.4. The integer t of Lemma 3.3(iv) can be easily computed using the exact sequence (3) to compute $c_1(\mathcal{E})$. One gets:

$$t = \sum_{i=1}^{m} p_i - \sum_{i=1}^{n} a_i + (n - m)(q + 2). \tag{7}$$

The next Theorem is the best we can do in connection with Problem 1.1, and implies Theorem 1.5 stated in the introduction.

Theorem 3.5. *Let M be a non-zero finite length graded R-module. Let $b := b(M)$ be the least degree of a second syzygy of M and let $q := q(M)$. Consider the following conditions:*

(i) $\mathbb{B}(M)$ *contains a maximal rank curve;*
(ii) $\mathbb{B}(M)$ *contains a maximal rank curve in every shift;*
(iii) $\mathbb{B}(M)$ *contains a minimal curve of maximal rank;*
(iv) *every minimal curve in $\mathbb{B}(M)$ has maximal rank.*
(v) $b \geq q + 1$.

Then we have: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) *and* (v) $\Rightarrow$ (i).

Proof. The equivalence of (i) to (iv) is an easy consequence of the Lazarsfeld-Rao Theorem and of [9], p. 68, Remarque 3.7 (see also [8], Theorem 1.4).

The last implication follows from Lemma 3.3(vii). □

The implication (i) $\Rightarrow$ (v) in Theorem 3.5 is false in general, as the following example shows:

Example 3.6. Let $M := R/(x, y, f, g)$ where f and g are two general forms of degree 2. It is easy to check that $b = q = 2$, whence in particular $b < q + 1$. However let D be the curve corresponding to the homogeneous ideal $(x, y) \cap (f, g)$, i.e., the disjoint union of the line $x = y = 0$ and of the complete intersection $f = g = 0$. Then $M_D = M$ and $s(D) = 3$ whence D has maximal rank. Then the implication (i) $\Rightarrow$ (v) of Theorem 3.5 is false.

We observe that the minimal curves in $\mathbb{B}(M)$ are double lines of genus -2 and Hartshorne-Rao module $M(1)$, and they have maximal rank, in agreement with Theorem 3.5.

Moreover we see that the construction of Lemma 3.3 produces a curve $C \in \mathbb{B}(M)$, with $M_C = M(-2)$, and not of maximal rank (by the same lemma).

Now we want to deal with Problems 1.2 and 1.3. Our goal is Theorem 3.9, which includes Theorem 1.6. We need some preparation.

Definition 3.7. Let M and q be as in 3.1. We say that M *has a nice tail* (resp. *a very nice tail*) if the map $M_{q-1} \to M_q$ induced by a general (resp. every) linear form is injective.

Proposition 3.8. *Let the notation be as in* 3.1 *and let C and t be as in Lemma* 3.3. *Assume that C has maximal rank (i.e., $q+1 \leq b \leq q+2$). Then we have:*

(i) $b = q+2$ *iff* $s(C) = t$. *Moreover in this case* $s(C) > \sigma(C)$;
(ii) *If* $b = q+1$ *then* $s(C) = t-1$. *Moreover:*
  (iia) $s(C) = \sigma(C)$ *if and only if M has a nice tail;*
  (iib) *If M has a very nice tail then* $s(C) = \sigma^*(C)$.

Proof. By Lemma 3.3 we get $r_0(C) = t-2 < s(C) \leq t$, $\sigma(C) \leq t-1$ and moreover $s(C) = t$ iff $b = q+2$. The conclusion follows easily by Remark 2.1. □

Theorem 3.9. *Let M be a non-zero finite length graded R-module and set $q := q(M)$ and $b := b(M)$. Assume that $b \geq q+1$. Then we have:*

(i) *There exists $D \in \mathbb{B}(M)$ of maximal rank, with $s(D) = \sigma(D)$ (resp. $s(D) = \sigma^*(D)$), if and only if M has a nice (resp. very nice) tail. Moreover D can be smooth and irreducible;*
(ii) *If $b = q+2$ every minimal curve $X \in \mathbb{B}(M)$ has maximal rank but $s(X) > \sigma(X)$. Hence no curve as in* (i) *can be minimal.*

Proof. Let $\mathcal{E}$ and t be as in Lemma 3.3 and set $\mathcal{F} := \mathcal{E}(q+2) \oplus \mathcal{O}(1)$. Then $\operatorname{rk}\mathcal{F} = n-m+1$, $c_1(\mathcal{F}) = t+1$, $\mathcal{F}$ is generated by global sections (since $\mathcal{E}$ is) and has no direct summand isomorphic to $\mathcal{O}$ (since $\mathcal{E}$ doesn't). Then by Lemma 3.2 there is a smooth curve D whose ideal fits into an exact sequence of the form

$$0 \to \mathcal{O}^{n-m} \to \mathcal{E}(q+2) \oplus \mathcal{O}(1) \to \mathcal{I}_D(t+1) \to 0.$$

By Lemma 3.3, we have $H^0(\mathcal{E}(q)) = 0$ (since $b \geq q+1$ by assumption) and $H^1_*(\mathcal{F})(-q-2) = H^1_*(\mathcal{E}) = M$. Then by the above exact sequence we have $s(D) = t$ and $M_D = M(-(t+1)+q+2)$ whence $r_o(D) = t-1 = s(D)-1$ and in particular D has maximal rank.

Moreover since $t \geq q+2$ by Lemma 3.3(ix) we have $r_a(D) = t-1-q > 0$ whence D is connected, hence integral. Since D is clearly non planar no linear form vanishes identically on any component of D. Now since $r_o(D) = s(D)-1$, (i) follows easily by Remark 2.1.

Now let $b = q+2$ and assume X minimal. Let C be the curve constructed in Lemma 3.3(iv). Since $b = q+2$ this curve is minimal (same lemma, item (vii)). Then by [9], p. 88, Théorème 4.3, we may assume that $X = C$. Then (ii) follows from Proposition 3.8. □

4. Cyclic Hartshorne-Rao modules and Rao functions of maximal rank curves

In this section we deal with Problem 1.4 and in particular we prove Theorem 1.7.

Since Problem 1.4 seems to be rather difficult, we will deal mainly with an intermediate step, namely:

Problem 4.1. To characterize the Rao functions of maximal rank curves C with cyclic Rao module and with the lifting property $s(C) = \sigma(C)$ (resp. $s(C) = \sigma^*(C)$).

So we assume $M = R/I$, where $I \neq R$ is a homogeneous irrelevant ideal. We keep the notation of 3.1. Observe that in this case $m = 1$, $a_1 \leq \cdots \leq a_n$ are the degrees of a minimal set of generators of I, and $b = b(M)$ is the least degree of a syzygy of I. We let $I = (f_1, \ldots, f_n)$, where $\deg f_i = a_i$. It is useful to take into account the diameter of M, which in this case coincides with the Buchsbaum index $k := k(M)$. Note that then $q = k - 1$.

We begin the case of Koszul modules (studied by Martin-Deschamps and Perrin in [9]). In this case the picture is complete.

Proposition 4.2. *Assume that M is a Koszul module (i.e., $n = 4$ and f_1, f_2, f_3, f_4 form a regular sequence). Then:*

(a) *$\mathbb{B}(M)$ contains a maximal rank curve if and only if $4 > a_2 + a_4$, if and only if $(a_1, a_2, a_3, a_4) \in \{(1,1,1,1), (1,1,1,2), (1,1,2,2)\}$;*

(b) *$\mathbb{B}(M)$ contains a maximal rank curve with $s(C) = \sigma(C)$ if and only if $(a_1, a_2, a_3, a_4) \in \{(1,1,1,1), (1,1,1,2)\}$;*

(c) *$\mathbb{B}(M)$ contains a maximal rank curve with $s(C) = \sigma^*(C)$ if and only if $(a_1, a_2, a_3, a_4) = (1,1,1,1)$.*

Proof. (a). By Theorem 3.5, $\mathbb{B}(M)$ contains a maximal rank curve if and only if $\mathbb{B}(M)$ contains a minimal curve of maximal rank. Let C_0 be the minimal curve associated to M. Set $\mu := \sup\{a_1 + a_4, a_2 + a_3\}$. According to [9] Ch. IV, Corollaire 6.7 and Remarques 6.8, $M_{C_0} = M(a_3 + a_4 - \mu)$ and $s(C_0) = \mu + a_1 - a_4$. Since $q(M) = a_1 + a_2 + a_3 + a_4 - 4$, we conclude that C_0 has maximal rank if and only if $\mu + a_1 - a_4 > \mu + a_1 + a_2 - 4$ or, equivalently, $4 > a_2 + a_4$ which proves what we want.

(b) (c). If $(a_1, a_2, a_3, a_4) \in \{(1,1,1,1), (1,1,1,2)\}$ the conclusion follows easily from (a) and Theorem 3.9. If $(a_1, a_2, a_3, a_4) = (1,1,2,2)$ the Rao function is $1, 2, 1, 0 \ldots$ (up to shift) and then $s(C) > \sigma(C)$ by Remark 2.1(iv) and (v). □

Now we study the easy case $I = \mathfrak{m}^k$. The module $M = R/\mathfrak{m}^k$ has the maximum Hilbert function among the cyclic modules with Buchsbaum index k, namely:

$$1, 4, \ldots, \binom{k+2}{3}, 0, \ldots.$$

Proposition 4.3. *Let $k > 0$ be an integer and let $M := R/\mathfrak{m}^k$. Then every minimal curve $C \in \mathbb{B}(M)$ has maximal rank and satisfies $s(C) > \sigma(C)$.*

Moreover there is a maximal rank curve $D \in \mathbb{B}(M)$ satisfying $s(D) = \sigma^*(D)$.

Proof. We have $M_i = R_i$ for $i < k$ and $M_i = 0$ for $i \geq k$. It follows that M has a very nice tail. Moreover $b = k + 1 = q + 2$. The conclusion follows from Theorem 3.9. □

From now on we fix k and we assume $I \neq \mathfrak{m}^k$. Let $h := \max\{j \mid a_j < k\}$ and set $J := (f_1, \ldots, f_h)$. Note that $h \geq 1$, that J is minimally generated by $f_1, \ldots, f_h$ and that $I = J + \mathfrak{m}^k$. Moreover we have:

$$(R/I)_i = \begin{cases} (R/J)_i & \text{for} \quad i < k \\ 0 & \text{for} \quad i \geq k \end{cases} \tag{8}$$

In particular the Hilbert function of R/I can be considered as the Hilbert function of R/J truncated at k.

Lemma 4.4. *Let $M = R/I$ with Buchsbaum index $k > 0$ and $I \neq \mathfrak{m}^k$. Let $I = J + \mathfrak{m}^k$ be a decomposition as above and denote by $b(R/J)$ the least degree of a syzygy of J. Then we have:*

(a) *if J is saturated (resp. no associated prime ideal of J contains linear forms) then M has a nice (resp. very nice) tail;*
(b) *If J is non-principal (i.e., $h > 1$) then*
  (b1) *$b(R/J) \geq b(M)$;*
  (b2) *$b(R/J) \geq k$ if and only if $b(M) \geq k$;*
  (b3) *$b(M) = k$ if and only if $b(R/J) = k$;*
(c) *If J is principal (namely $h = 1$) then $b(M) = k + 1$.*

Proof. (a). The assumption on J implies that a general linear form (resp. every linear form) is R/J-regular, hence in particular it induces an injective linear map $(R/J)_{k-2} \to (R/J)_{k-1}$.

(b1). Obvious.

(b2). Assume $b(M) < k$. Then a syzygy of I of degree $b(M)$ must be a syzygy of J, whence $b(M) \geq b(R/J)$. But by (b1) this implies $b(R/J) = b(M) < k$. The converse is clear.

(b3) If J has no syzygy of degree k, then some of the minimal generators of I of degree k must belong to J, a contradiction. Then $b(R/J) \leq k$ and the conclusion follows from (b1). The converse is clear.

(c) Left to the reader. □

Now we want to deal with a Rao function which is maximal except at the last place, namely:

$$1, 4, 10, \ldots, \binom{k+1}{3}, \binom{k+2}{3} - \epsilon, 0 \to \tag{9}$$

where $\binom{k+2}{3} > \epsilon > 0$.

To get this function we must have $I = J + \mathfrak{m}^k$, with $J = (f_1, \ldots, f_\epsilon)$, where $f_1, \ldots, f_\epsilon$ are linearly independent forms of degree $k - 1$. We are going to study which cases are possible.

Lemma 4.5. *Let the notation be as above and assume $k \geq 2$. Then we have:*

(i) *For every $\epsilon < \binom{k+2}{3}$ there is a cyclic module M with Hilbert function (9). Any such module satisfies $k+1 \geq b \geq k$;*

(ii) *Let M be as in* (i). *If $b = k+1$ then $4\epsilon \leq \binom{k+3}{3}$. Conversely if $4\epsilon \leq \binom{k+3}{3}$ and $f_1, \ldots, f_\epsilon$ are sufficiently general then $b = k+1$;*

(iii) *There exists M as in* (i) *with a nice tail if and only if $\epsilon \leq \binom{k+1}{2}$;*

(iv) *There exists M as in* (i) *with a very nice tail if and only if $\epsilon \leq \binom{k+1}{2} - 3$.*

Proof. (i) We must take $M = R/I$, where I is as above. The minimal generators of I have degree $\geq k-1$, whence the syzygies have degree $\geq k$, whence $b \geq k$. The other inequality is always satisfied.

(ii) If $b(M) = k+1$ then $b(R/J) \geq k+1$, that is J has no syzygy of degree k. It follows that $R_1\langle f_1, \ldots f_\epsilon\rangle \subseteq R_k$ has maximal dimension 4ϵ, whence $4\epsilon \leq \binom{k+3}{3}$.

Conversely if $f_1, \ldots, f_\epsilon$ are general they have no syzygy of degree k by [7], Theorem 1. Then $b = k+1$ by Lemma 4.4(b3).

(iii) Let $\ell \in R_1$ be any linear form. Since $\epsilon \leq \binom{k+1}{2} = \binom{k+2}{3} - \binom{k+1}{3}$ there are $f_1, \ldots f_\epsilon \in R_{k-1}$ linearly independent modulo ℓR_{k-2}. Let $I := (f_1, \ldots f_\epsilon) + \mathfrak{m}^k$. If $M := R/I$ it is then clear that M satisfies (i) and multiplication by ℓ is an injective linear map $M_{k-2} \to M_{k-1}$. Hence by semicontinuity the same is true for the map induced by a general linear form.

The converse is clear.

(iv) Assume that M exists. Then by the bilinear Lemma (see the proof of 2.2) we have $\binom{k+2}{3} - \epsilon = \dim(M_{k-1}) \geq \dim(M_{k-2}) + 3 = \binom{k+1}{3} + 3$, whence $\epsilon \leq \binom{k+1}{2} - 3$.

Conversely assume that $\epsilon \leq \binom{k+1}{2} - 3$ (whence $\epsilon \leq \binom{k+2}{3}$, so that (i) applies). Consider the morphism

$$\varphi : \mathbb{P}(R_{k-2}) \times \mathbb{P}(R_1) \to \mathbb{P}(R_{k-1})$$

induced by multiplication. Then

$$\dim(\mathrm{Im}(\varphi)) \leq \dim(\mathbb{P}(R_{k-2})) + \dim(\mathbb{P}(R_1)) = \binom{k+1}{3} + 2,$$

whence $\mathrm{codim}(\mathrm{Im}(\varphi)) \geq \epsilon$. It follows that a general linear variety $L \subseteq \mathbb{P}(R_{k-1})$ of dimension $\epsilon - 1$ does not intersect $\mathrm{Im}(\varphi)$. This easily implies that the ϵ-dimensional linear subspace V of R_{k-1} corresponding to L satisfies $V \cap \ell R_{k-2} = \{0\}$ for every linear form ℓ, whence any linear form induces an injective map $R_{k-2} \to R_{k-1}/V$.

Then it is sufficient to take $I := (f_1, \ldots, f_\epsilon) + \mathfrak{m}^k$, where $f_1, \ldots, f_\epsilon$ form a basis of V. □

Corollary 4.6. *Let ϵ and k be two integers. Then*

(i) *If $0 < \epsilon \leq \binom{k+1}{2}$ there exists a maximal rank k-Buchsbaum curve C with Rao function* (9) (*up to shift*) *and satisfying $s(C) = \sigma(C)$.*

(ii) *If $0 < \epsilon \leq \binom{k+1}{2} - 3$ there exists a maximal rank k-Buchsbaum curve C with Rao function* (9) (*up to shift*) *and satisfying $s(C) = \sigma^*(C)$.*

(iii) *If moreover* $4\epsilon \leq \binom{k+3}{3}$ *and* M *is as in Lemma* 4.5 *with general* $(f_1, \ldots, f_\epsilon)$ *then no minimal curve* $C \in \mathbb{B}(M)$ *satisfies* $s(C) = \sigma(C)$ (*although it has maximal rank and Rao function* (9)).

Proof. (i) and (ii) follow by Lemma 4.5 (i), (iii), (iv) and by Theorem 3.9.

(iii) We have $b = k + 1 = q + 2$ by Lemma 4.5(ii), and the conclusion follows by Theorem 3.9(ii). □

As a consequence of the above corollary we can give the following complement to Lemma 2.2.

Example 4.7. Item (iii) of Lemma 2.2 is false under the weaker assumption $s(C) = \sigma(C)$, in other words the given statement is the best possible. Indeed for every $k > 2$ there is a maximal rank k-Buchsbaum curve C with $s(C) = \sigma(C)$ whose Rao function is (up to shift)

$$1, 4, 10, \ldots, \binom{k+1}{3}, \binom{k+1}{3}, 0 \to \tag{10}$$

This follows from Corollary 4.6(i), with $\epsilon = \binom{k+1}{2}$.

Note that then we must have $\sigma(C) > \sigma^*(C)$.

Now we want to study smaller Rao functions. The first case is the following:

Proposition 4.8. *Let* $M = R/I$ *where* $I := (f) + \mathfrak{m}^k \subset R$, *with* $0 < \deg(f) < k$. *Then there is a maximal rank curve* $C \in \mathbb{B}(M)$ *satisfying* $s(C) = \sigma(C)$. *If moreover* f *has no linear factors there is* C *as above satisfying* $s(C) = \sigma^*(C)$.

In both cases C *cannot be minimal.*

Proof. It follows by Lemma 4.4(a)(c) and by Theorem 3.9. □

Remark 4.9. Let M and C be as in Proposition 4.8 and put $a = a_1 = \deg f$. Then by (8) and an easy computation we have that the Hilbert function of M (hence ρ_C up to shift) is the following:

$$\dim_K(M_i) = \begin{cases} 0 & \text{if} \quad i < 0 \\ \binom{i+3}{3} & \text{if} \quad 0 \leq i < a \\ \binom{i+3}{3} - \binom{i-a+3}{3} & \text{if} \quad a \leq i < k \\ 0 & \text{if} \quad i \geq k \end{cases}.$$

Corollary 4.10. *For every* $k \geq 2$ *and every* ϵ *such that* $0 \leq \epsilon < \binom{k+1}{2}$ *there is a maximal rank curve whose Rao function is, up to shift:*

$$1, 3, 6, \ldots, \binom{k}{2}, \binom{k+1}{2} - \epsilon, 0 \ldots$$

Proof. The case $\epsilon = 0$ follows from Proposition 4.8 with $\deg f = 1$ and by Remark 4.9. For $\epsilon > 0$, it is enough to consider $I := (f, g_1, \ldots, g_\epsilon) + \mathfrak{m}^k \subset R$ where f is a form of degree 1 and $g_1, \ldots, g_\epsilon$ are linearly independent forms of degree $k - 1$ in $R/(f)$. □

Let now C be a k-Buchsbaum curve of maximal rank containing no planar subcurves and satisfying $s(C) = \sigma^*(C)$. Then by Lemma 2.2 ρ_C is bounded below (up to shift) by:

$$1, 4, 7, \ldots, 1 + 3(k-1), 0, \ldots \tag{11}$$

We want to understand to which extent this lower bound can be attained.

Proposition 4.11. *Let f_1, f_2, f_3 be linearly independent forms of degree 2 and let $I := (f_1, f_2, f_3) + \mathfrak{m}^k$ and set $M := R/I$. Then there is a maximal rank curve $C \in \mathbb{B}(M)$ if one of the following occurs:*

(i) *$k = 3$. In this case ρ_C is $1, 4, 7, 0, \ldots$ (up to shift);*
(ii) *$k = 4$ and f_1, f_2, f_3 form a regular sequence. In this case ρ_C is $1, 4, 7, 8, 0, \ldots$ (up to shift).*

Moreover if $k = 3$ (resp. $k = 4$) then there exists C with the further property $s(C) = \sigma^*(C)$ (resp. $s(C) = \sigma(C)$).

Proof. It is easy to see, using Lemma 4.4, that $b(M) \geq 3$ and that $b(M) = 4$ if f_1, f_2, f_3 form a regular sequence. Then (i) and (ii) follow by Theorem 3.5 and an easy calculation for the Rao functions.

To prove the last statement just observe that by a suitable choice of f_1, f_2, f_3 Lemma 4.4(a) applies, whence the conclusion by Theorem 3.9. □

Now we want to give a few cases in which non-cyclic Hartshorne-Rao module are applied.

Our first result in this direction shows that using non-cyclic modules some other Rao function can be obtained with respect to Proposition 4.11.

Proposition 4.12. *For every $r \geq 1$ there is a 4-Buchsbaum curve of maximal rank with Rao function $1, 4, 7, r, 0 \ldots$ (up to shift).*

Proof. Assume first that $r \geq 8$. Let f_1, f_2, f_3 be a regular sequence of forms of degree 2, and set $J := (f_1, f_2, f_3)$. Let $N := R/J + \mathfrak{m}^4$, $P := (R/\mathfrak{m})^{r-8}$ and $M := N \oplus P(-3)$. Then $k(M) = 4$, $q(M) = 3$, $b(N) = 4$, $b(P(-3)) = 5$, whence $b(M) = 4 \geq q(M)+1$. Moreover since the Hilbert function of R/J is $1, 4, 7, 8, 8, \ldots$, it is clear that the Hilbert function of M is $1, 4, 7, r, 0, \ldots$. Then by Theorem 3.5 $\mathbb{B}(M)$ contains a curve of maximal rank, which has the required properties.

Assume now $1 \leq r \leq 8$. Let J be as before, and recall that $\dim_K(R/J)_3 = 8$. Choose $g_1, \ldots, g_{8-r} \in R_3$ linearly independent modulo J_3 and set $I := J + (g_1, \ldots, g_{8-r}) + \mathfrak{m}^4$ and $M := R/I$. Then M has the required Hilbert function and moreover $b(M) = 4 = k(M) = q(M) + 1$. The conclusion follows then by Theorem 3.5. □

Remark 4.13. We don't know if for every $k > 3$ there is a maximal rank curve with Rao function (11) (up to shift). It is clear that if we want also a cyclic Hartshorne-Rao module the only possibility is to start with M as in Proposition 4.11 where the ideal (f_1, f_2, f_3) has a linear syzygy. Then the construction of Lemma 3.3 works only for $k \leq 3$. In view of Propositions 4.11 and 4.12 the first cases which are unknown to us are: $k = 4$ and M cyclic, and $k = 5$ (any M).

We conclude with two results concerning k-Buchsbaum curves of maximal rank.

Proposition 4.14.

(a) *Let C be a 1-Buchsbaum curve of maximal rank with the lifting property $s(C) = \sigma(C)$. Then $M_C \cong (R/\mathfrak{m})^a(-s(C)+1)$ for some $a > 0$ and the Rao function of C is*

$$\rho_C(j) = \begin{cases} 0 & \text{if} \quad j \neq s(C)-1 \\ a & \text{if} \quad j = s(C)-1. \end{cases}$$

(b) *Let $M \cong (R/\mathfrak{m})^a$ with $a > 0$. Then we have:*
   (i) *$\mathbb{B}(M)$ contains maximal rank curves in every shift;*
   (ii) *Every minimal curve $X \in \mathbb{B}(M)$ satisfies $s(X) > \sigma(X)$;*
   (iii) *There is a smooth connected curve $Y \in \mathbb{B}(M)$ having maximal rank and satisfying $s(Y) = \sigma(Y) = \sigma^*(Y)$.*

Proof. (a) By Proposition 2.3 we have $\operatorname{diam}(M_C) = 1$. The conclusion follows by Remark 2.1.

(b) From the Koszul resolution of $R/\mathfrak{m}$ we get easily $b(M) = 2 = q(M)+2$. Then Theorem 3.5 implies (i) and Theorem 3.9 implies (ii) and (iii). □

Our last example is related to Proposition 2.3.

Example 4.15. For every $k \geq 1$ there is a maximal rank k-Buchsbaum curve C such that $\operatorname{diam}(M_C) = k+1$.

Indeed let $M := N \oplus N(-1)$ where $N = R/\mathfrak{m}^k$. Clearly $q(M) = k$ and $b(M) = b(N) = k+1 = q(M)+1$. Then there is a maximal rank curve $C \in \mathbb{B}(M)$ by Theorem 3.5. Obviously C is k-Buchsbaum and $\operatorname{diam}(M_C) = \operatorname{diam}(M) = k+1$.

Observe that the inequality $s(C) > \sigma(C)$ forced by Proposition 2.3 is confirmed by Remark 2.1, because by construction every linear map $M_{q(M)-1} \to M_{q(M)}$ induced by a linear form is non-injective.

Are there examples of this sort with an *indecomposable* M?

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Doubling Rational Normal Curves

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Abstract. In this paper, we study double structures supported on rational normal curves. After recalling the general construction of double structures supported on a smooth curve described in [11], we specialize it to double structures on rational normal curves. To every double structure we associate a triple of integers $(2r, g, n)$ where r is the degree of the support, $n \geq r$ is the dimension of the projective space containing the double curve, and g is the arithmetic genus of the double curve. We compute also some numerical invariants of the constructed curves, and we show that the family of double structures with a given triple $(2r, g, n)$ is irreducible. Furthermore, we prove that the general double curve in the families associated to $(2r, r+1, r)$ and $(2r, 1, 2r-1)$ is arithmetically Gorenstein. Finally, we prove that the closure of the locus containing double conics of genus $g \leq -2$ form an irreducible component of the corresponding Hilbert scheme, and that the general double conic is a smooth point of that component. Moreover, we prove that the general double conic in $\mathbb{P}^3$ of arbitrary genus is a smooth point of the corresponding Hilbert scheme.

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1. Introduction

Non-reduced projective curves arise naturally when one tries to classify smooth curves, where a projective curve is a dimension 1 projective scheme without embedded or isolated 0-dimensional components. In fact, two of the main tools to classify projective curves are liaison theory and deformation theory.

Given two curves C and D embedded in the projective space $\mathbb{P}^n$, we say that they are geometrically linked if they have no common component and their union is an arithmetically Gorenstein curve. More than the geometric links, a modern

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treatment of the theory takes as its base the algebraic link where two curves are algebraically linked via the arithmetically Gorenstein curve X if $I_X : I_C = I_D$ and $I_X : I_D = I_C$, where I_C, I_D, I_X are the saturated ideals that define the curves C, D, X, respectively, in the projective space $\mathbb{P}^n$. If C and D have no common irreducible component, the two definitions agree. Liaison theory and even liaison theory are the study of the equivalence classes of the equivalence relation generated by the direct link, and by an even number of direct links, respectively. In $\mathbb{P}^3$, a curve is arithmetically Gorenstein if, and only if, it is the complete intersection of two algebraic surfaces. A pioneer in the study of this theory for curves in $\mathbb{P}^3$ was F. Gaeta (see [13]). In the quoted paper, he proved that every arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ is in the equivalence class of a line. More in general, every curve sits in an equivalence class, and, for curves in $\mathbb{P}^3$, it is known that every curve in a biliaison class can be obtained from the curves of minimal degree in the class via a rather explicit algorithm. This property is known as Lazarsfeld-Rao property ([23], Definition 5.4.2), and it was proved in ([22], Ch. IV, Theorem 5.1). The existence of minimal curves and their construction is proved in ([22], Ch. IV, Proposition 4.1, and Theorem 4.3). In [1], the authors proved the Lazarsfeld-Rao property for the curves in the same biliaison class, without the explicit construction of the minimal curves. The minimal arithmetically Cohen-Macaulay curves are the lines. Hence, the Lazarsfeld-Rao property can be seen as a generalization of Gaeta's work. Also if one wants to study smooth curves, the minimal curves in the biliaison class can have quite bad properties, e.g., they can be non-reduced, or they can have a large number of irreducible components. Moreover, the minimal curves in a biliaison class form an irreducible family of curves with fixed degree and arithmetic genus. Today, it is not known if the equivalence classes of curves in $\mathbb{P}^n$ have the same properties as those in $\mathbb{P}^3$ (see [25], [28], [8], [16] for evidence both ways).

To study the properties of smooth curves, one can also try to deform the smooth curve to a limit curve and investigate the properties one is interested in on the limit curve. If those properties are shared by the limit curve and the deformation behaves well with respect to the considered properties, then the general curve shares the same properties of the limit curve. Often, the limit curves are non-reduced curves. In the papers [12], [3], [10], the authors study Green's conjecture concerning the free resolution of a canonical curve by reducing it to the study of a similar conjecture for double structures on $\mathbb{P}^1$ called ribbons.

Both described approaches lead to the study of families of curves. The universal family of curves of fixed degree d and arithmetic genus g is the Hilbert scheme $\mathcal{H}ilb_{dt+1-g}(\mathbb{P}^n)$, where, for us, $\mathcal{H}ilb_{dt+1-g}(\mathbb{P}^n)$ is the open locus of the full Hilbert scheme corresponding to locally Cohen-Macaulay 1-dimensional schemes, i.e., corresponding to curves. Since A. Grothendieck proved its existence in [14], the study of the properties of the Hilbert scheme attracted many researchers. In spite of their efforts, only a few properties are known, such as the connectedness of the full Hilbert scheme proved by R. Hartshorne in [15]. A current trend of

research tries to generalize Hartshorne's result on connectedness to the Hilbert scheme of curves. For partial results on the problem, see, for example, [18], [27]. In studying Hilbert schemes, a chance is to relate the local properties of a point on the Hilbert scheme, e.g., its smoothness on $\mathcal{H}ilb_{dt+1-g}(\mathbb{P}^n)$, and the global properties of the curve embedded in $\mathbb{P}^n$. With abuse of notation, we denote C both the curve in $\mathbb{P}^n$ and the corresponding point on $\mathcal{H}ilb_{dt+1-g}(\mathbb{P}^n)$. It is well known that the tangent space to $\mathcal{H}ilb_{dt+1-g}(\mathbb{P}^n)$ at a point C can be identified with $H^0(C,\mathcal{N}_C)$ where $\mathcal{N}_C$ is the normal sheaf of the curve C as subscheme of $\mathbb{P}^n$ ([29], Theorem 4.3.5). However, both $\mathcal{N}_C$ and its degree 0 global sections are far from being well understood for an arbitrary curve C.

Both in liaison and biliaison theory, and in deformation theory, one has often to consider non-reduced curves. The first general construction for non-reduced curves was given by D. Ferrand in [11], where the author constructs a double structure on a smooth curve $C \subset \mathbb{P}^n$. The construction was investigated in [6], and generalized in [4] to multiple structures on a smooth support. In the last quoted paper, the authors present a filtration of a multiple structure X on a smooth support C via multiple structures with smaller multiplicity on the same curve C. Moreover, they relate the properties of X to the ones of the curves in the filtration. A different filtration was proposed in [20]. When the first multiple structure in either filtration has multiplicity 2 at every point, then it comes from Ferrand's construction. In this sense, double structures are the first step in studying multiple structures on a smooth curve C. Because of the previous discussion, it is interesting to understand if double structures on curves form irreducible families, if they fill irreducible components of the Hilbert scheme (if so, they cannot be limit curves of smooth curves), and if, among them, there are curves with properties that are preserved under generalization, such as the property of being arithmetically Gorenstein. In the papers [25], [24], [26], the authors study the stated problems for double structures on lines, and more generally for a multiple structure X on a line $L \subset \mathbb{P}^n$ satisfying the condition $I_L^2 \subseteq I_X \subseteq I_L$, called ropes in the literature. In the present paper, we address the same problems for double structures supported on the most natural generalization of a line, i.e., a rational normal curve. In [21], the author considers double structures on rational normal curves, but he is interested in the ones with linear resolution, a class of curves different from the ones we investigate.

The plan of the paper is the following. In Section 2, we recall Ferrand's construction of double structures on smooth curves, and we specialize it to construct double structures on rational normal curves. To set notation and for further use in the paper, we recall some known facts about rational normal curves. Moreover, we prove that we obtain the saturated ideal of the double structure directly from the construction, and we compute the Hilbert polynomial (and hence the arithmetic genus) of the double structure in terms of the numerical data of the construction. Finally, we compute the dimension of the irreducible family of double rational normal curves of given genus. In Section 3, we compute the Hartshorne-Rao function $h^1\mathcal{I}_X(j)$ of such a doubling X, for $j \neq 2$, and we bound $h^1\mathcal{I}_X(2)$. To get the results,

we give also some results about the ideal sheaf $\mathcal{I}_C^2$ where $C = X_{\mathrm{red}}$ is the rational normal curve support of X. Probably, the results we prove on $\mathcal{I}_C^2$ are folklore, but we did not find references in literature. In Section 4, we prove that, among the double curves we are studying, we can obtain arithmetically Gorenstein curves. In more detail, it happens in two cases: if X has genus $r+1$ in $\mathbb{P}^r$, i.e., X has degree and genus of a canonical curve in $\mathbb{P}^r$, and if X has genus 1 in $\mathbb{P}^{2r-1}$, i.e., X has degree and genus of a non-degenerate normal elliptic curve in $\mathbb{P}^{2r-1}$. The former curves were originally studied in [3] to understand Green' s conjecture on the free resolution of canonical curves. In the same paper, the authors, together with J. Harris, prove that the considered double structures on rational normal curves are smooth points of the component of the Hilbert scheme containing canonical curves ([3], Theorem 6.1). In the last section of the paper, we study the local properties of $H(4,g,n)$, and we show that, if $g \leq -2$, then $H(4,g,n)$ is open in a generically smooth irreducible component of the Hilbert scheme $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^n)$. Moreover, we also prove that the general double conic is a smooth point of $H(4,g,3)$ with $g \geq -1$, that $H(4,g,3)$ is not an irreducible component of $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^3)$, and we exhibit the general element of the irreducible component $\overline{H(4,g,3)}$ containing $H(4,g,3)$. The results in this section partially complete the ones in [27]. In fact, in [27], the authors prove that the Hilbert scheme $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^3)$ is connected, but do not study its local properties. By the way, in [27], the double conics are studied as particular curves contained in a double plane, curves studied in [18], and so their construction and their properties are not considered. Finally, in [5], the authors proved that double conics in $\mathbb{P}^3_{\mathbb{C}}$ of genus -5 are smooth points of the Hilbert scheme $\mathcal{H}ilb_{4t+6}(\mathbb{P}^3_{\mathbb{C}})$.

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2. Construction of double rational normal curves

Let K be an algebraically closed field of characteristic 0 and let $\mathbb{P}^n$ be the n-dimensional projective space over K defined as $\mathbb{P}^n = \mathrm{Proj}(R := K[x_0, \ldots, x_n])$. If $X \subset \mathbb{P}^n$ is a closed subscheme, we define $\mathcal{I}_X$ its ideal sheaf in $\mathcal{O}_{\mathbb{P}^n}$ and we define the normal sheaf $\mathcal{N}_X$ of X in $\mathbb{P}^n$ as $\mathcal{N}_X = \mathcal{H}om_{\mathbb{P}^n}(\mathcal{I}_X, \mathcal{O}_X) = \mathcal{H}om_X(\frac{\mathcal{I}_X}{\mathcal{I}_X^2}, \mathcal{O}_X)$. The saturated ideal of X is the ideal $I_X = \oplus_{j\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{I}_X(j)) \subseteq R$, and it is a homogeneous ideal. The homogeneous coordinate ring of X is defined as $R_X = R/I_X$, and it is naturally graded over $\mathbb{Z}$. The Hilbert function of X is then the function defined as $h_X(j) = \dim_K(R_X)_j$, degree j part of R_X, for $j \in \mathbb{Z}$. Finally, it is known that there exists a polynomial $P(t) \in \mathbb{Q}[t]$, called Hilbert polynomial of X, that verifies $P(t) = h_X(t)$ for $t \in \mathbb{Z}, t \gg 0$. The degree of $P(t)$ is the dimension of X. If X is a locally Cohen-Macaulay curve, then $P(t) = dt + 1 - g$ for some integers d, g, referred to as degree and arithmetic genus of X, respectively.

Given a smooth curve $C \subset \mathbb{P}^n$, there is a well-known method, due to D. Ferrand (see [11]), to construct a non-reduced curve X, having C as support, and multiplicity 2 at each point. X is called a *doubling* of C.

Ferrand's method works as follows.

Let $\mathcal{I}_C$ be the ideal sheaf of $C \subset \mathbb{P}^n$, and let $\mathcal{I}_C/\mathcal{I}_C^2$ be its conormal sheaf. If $\mathcal{L}$ is a line bundle on C, every surjective morphism $\mu : \frac{\mathcal{I}_C}{\mathcal{I}_C^2} \to \mathcal{L}$ gives a doubling X of C defined by the ideal sheaf $\mathcal{I}_X$ such that $\ker(\mu) = \mathcal{I}_X/\mathcal{I}_C^2$. The curves C, X and the line bundle $\mathcal{L}$ are related each other via the exact sequences

$$0 \to \mathcal{I}_X \to \mathcal{I}_C \to \mathcal{L} \to 0 \tag{2.1}$$

and

$$0 \to \mathcal{L} \to \mathcal{O}_X \to \mathcal{O}_C \to 0. \tag{2.2}$$

Moreover, X is a locally Cohen-Macaulay curve and its dualizing sheaf satisfies $\omega_X|C = \mathcal{L}^{-1}$.

We are interested in studying doublings of rational normal curves, where, for us, a rational normal curve C of degree r is the image of

$$\mathbb{P}^1 \xrightarrow{v_r} \mathbb{P}^r \longrightarrow \mathbb{P}^n$$

where v_r is the Veronese embedding and the second map is a linear embedding of $\mathbb{P}^r$ in $\mathbb{P}^n$ with $r \leq n$. To make effective Ferrand's construction in our case, we recall some known results about rational normal curves, and fix some notation.

Let $\mathbb{P}^r \cong L = V(x_{r+1}, \ldots, x_n) \subseteq \mathbb{P}^n = \mathrm{Proj}(R := K[x_0, \ldots, x_n])$ and let $C \subset L$ be the rational normal curve defined by the 2×2 minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & \ldots & x_{r-1} \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}.$$

In L the resolution of the saturated ideal $I_{C,L} \subset S := K[x_0, \ldots, x_r]$ of C is described by the Eagon-Northcott complex and it is

$$0 \to \wedge^r F \otimes S_{r-2}(G)^* \otimes \wedge^2 G^* \to \wedge^{r-1} F \otimes S_{r-3}(G)^* \otimes \wedge^2 G^* \to \ldots$$
$$\cdots \to \wedge^3 F \otimes S_1(G)^* \otimes \wedge^2 G^* \to \wedge^2 F \otimes S_0(G)^* \otimes \wedge^2 G^* \to \wedge^0 F \otimes S_0(G) \to 0$$

where $F = S^r(-1), G = S^2$ and $\varphi_A : F \to G$ is defined by the matrix A.

Remark 2.1. Because of the definition of F and G we have that the complex ends as follows

$$\cdots \to \wedge^3 F \otimes G^* \xrightarrow{\varepsilon} \wedge^2 F \xrightarrow{\phi_A} S \to S/I_{C,L} \to 0,$$

where ϕ_A is defined via the 2×2 minors of A.

Let $e_1, \ldots, e_r$ be the canonical basis of F and let f_1, f_2 be the canonical basis of G^*. Then, the map ε is defined as

$$\varepsilon(e_i \wedge e_j \wedge e_h \otimes f_k) = x_{i-2+k} e_j \wedge e_h - x_{j-2+k} e_i \wedge e_h + x_{h-2+k} e_i \wedge e_j$$

for every $1 \leq i < j < h \leq r$, $k = 1, 2$ (see, e.g., A2.6.1 in [9]).

Now, we compute the resolution of the saturated ideal $I_C \subset R$ of C. Of course, $I_C = I^e_{C,L} + I_L = I^e_{C,L} + \langle x_{r+1}, \dots, x_n \rangle$, where $I^e_{C,L}$ is the extension of $I_{C,L}$ via the natural inclusion $S \hookrightarrow R$.

To get a minimal free resolution of $I^e_{C,L}$ it suffices to tensorise by $\otimes_S R$ the minimal free resolution of $I_{C,L}$. To simplify notation, we set $P_i = \wedge^{i+1} F \otimes_S S_{i-1}(G)^* \otimes_S \wedge^2 G^* \otimes_S R$, for $i = 1, \dots, r-1$. Hence, the minimal free resolution of $I^e_{C,L}$ is equal to

$$0 \to P_{r-1} \xrightarrow{\varepsilon_{r-1}} P_{r-2} \xrightarrow{\varepsilon_{r-2}} \cdots \xrightarrow{\varepsilon_3} P_2 \xrightarrow{\varepsilon_2} P_1 \xrightarrow{\varepsilon_1} I^e_{C,L} \to 0, \tag{2.3}$$

where the maps are obtained by tensorising the maps of the minimal free resolution of $I_{C,L}$ times the identity of R.

The minimal free resolution of I_L is given by the Koszul complex over x_{r+1}, $\dots, x_n$. Let $Q = R^{n-r}(-1)$ with canonical basis $e_{r+1}, \dots, e_n$ and let $\delta : Q \to I_L$ be defined as $\delta(e_i) = x_i, i = r+1, \dots, n$. If we set $Q_i = \wedge^i Q$ then the minimal free resolution of I_L is equal to

$$0 \to Q_{n-r} \xrightarrow{\delta_{n-r}} Q_{n-r-1} \xrightarrow{\delta_{n-r-1}} \cdots \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \xrightarrow{\delta} I_L \to 0 \tag{2.4}$$

where $\delta_i = \wedge^i \delta$.

Given the two resolutions (2.3) and (2.4) above, we can compute their tensor product (for the definition and details, see [9], §17.3), and we get

$$0 \to N_{n-1} \longrightarrow N_{n-2} \longrightarrow \dots \longrightarrow N_2 \longrightarrow N_1 \tag{2.5}$$

where $N_i = \oplus_{j+k=i+1} P_j \otimes_R Q_k$.

Lemma 2.2. *The complex* (2.5) *is a minimal free resolution of* $I^e_{C,L} \cap I_L$.

Proof. Since N_i is isomorphic to $R^{\beta_i}(-i-2)$ with $\beta_i = \sum_{j+k=i+1} j \binom{r}{j+1} \binom{n-r}{k}$, as computed from its definition, no addendum can be cancelled because of the shifts. It follows that if the complex (2.5) is a resolution of $I^e_{C,L} \cap I_L$ then it is its minimal free resolution.

At first, we prove that $I^e_{C,L} \cap \langle x_{r+1}, \dots, x_l \rangle = I^e_{C,L} \cdot \langle x_{r+1}, \dots, x_l \rangle$ for every $l = r+1, \dots, n$. In fact, if $f \in I^e_{C,L} \cap \langle x_{r+1}, \dots, x_l \rangle$ and $I^e_{C,L} = \langle g_1, \dots, g_t \rangle$, then there exist $h_1, \dots, h_t \in R$ such that $f = h_1 g_1 + \dots h_t g_t \in \langle x_{r+1}, \dots, x_l \rangle$. For each $i = 1, \dots, t$ there exist $h'_i \in K[x_0, \dots, x_r, x_{l+1}, \dots, x_n]$ and $h''_i \in \langle x_{r+1}, \dots, x_l \rangle$, both unique, such that $h_i = h'_i + h''_i$. Hence, $h'_1 g_1 + \dots + h'_t g_t \in \langle x_{r+1}, \dots, x_l \rangle$ and so it is equal to 0, because the variables $x_{r+1}, \dots, x_l$ appear neither in the g_i's nor in the h'_j's. Then, we have $f = h''_1 g_1 + \dots + h''_t g_t \in I^e_{C,L} \cdot \langle x_{r+1}, \dots, x_l \rangle$.

To prove that the complex (2.5) is a resolution of $I^e_{C,L} \cap I_L$ we use induction on the number of generators of I_L.

If $I_L = \langle x_{r+1} \rangle$, then $I^e_{C,L}(-1) \xrightarrow{\cdot x_{r+1}} I^e_{C,L} \cdot I_L$ is a degree 0 isomorphism, and so the resolution of $I^e_{C,L} \cdot I_L$ is the one of $I^e_{C,L}$ shifted by -1. In this case, $0 \to R(-1) \xrightarrow{\cdot x_{r+1}} I_L \to 0$ is the resolution of I_L. Thus, the complex (2.5) is equal to (2.3) tensorised by $\otimes_R R(-1)$ and so it is the resolution of $I^e_{C,L} \cdot I_L$.

Assume now that $I_L = \langle x_{r+1}, \ldots, x_n \rangle$ and that the statement holds for $I'_L = \langle x_{r+1}, \ldots, x_{n-1} \rangle$.

The two ideals $I^e_{C,L} \cdot I_L$ and $I^e_{C,L} \cdot I'_L + I^e_{C,L} \cdot \langle x_n \rangle$ are equal. Moreover, $I^e_{C,L} \cdot I'_L \cap I^e_{C,L} \cdot \langle x_n \rangle$ is equal to $I^e_{C,L} \cdot I'_L \cdot \langle x_n \rangle \cong I^e_{C,L} \cdot I'_L(-1)$. In fact, let $f \in I^e_{C,L}$ and assume that $x_n f \in I^e_{C,L} \cdot I'_l$. Then, $x_n f \in I^e_{C,L}$ and $x_n f \in I'_L$. But both the ideals are prime and x_n belongs neither to $I^e_{C,L}$ nor to I'_L. Hence, $f \in I^e_{C,L} \cap I'_L = I_{C,L} \cdot I'_L$ and the statement follows because the converse inclusion is evident. We have then the following short exact sequence

$$0 \to I^e_{C,L} \cdot I'_L(-1) \longrightarrow I^e_{C,L} \cdot I'_L \oplus I^e_{C,L} \cdot \langle x_n \rangle \longrightarrow I^e_{C,L} \cdot I_L \to 0$$

and the claim follows by applying the mapping cone procedure. □

Proposition 2.3. *With the same notation as before, the minimal free resolution of* I_C *is*

$$0 \to N_{n-1} \xrightarrow{\varepsilon'_{n-1}} \cdots \xrightarrow{\varepsilon'_3} P_2 \oplus Q_2 \oplus N_1 \xrightarrow{\varepsilon'_2} P_1 \oplus Q_1 \xrightarrow{\varepsilon'_1} I_C \to 0$$

where $\varepsilon'_2 : N_1 = P_1 \otimes Q_1 \longrightarrow P_1 \oplus Q_1$ *is defined as*

$$\begin{pmatrix} -x_{r+1} id_{P_1} & \cdots & -x_n id_{P_1} \\ \varepsilon_1 & \cdots & \varepsilon_1 \end{pmatrix}.$$

Proof. The ideal I_C is equal to $I^e_{C,L} + I_L$. Hence, we have the short exact sequence

$$0 \to I^e_{C,L} \cap I_L \longrightarrow I^e_{C,L} \oplus I_L \longrightarrow I_C \to 0.$$

By applying the mapping cone procedure, we get a free resolution of I_C that has the shape of our claim. The minimality of the resolution follows because a cancellation takes place in the resolution only if a free addendum of N_i splits from the map $N_i \to P_i \oplus Q_i$. This cannot happen because $N_i \cong R^{\beta_i}(-i-2)$, $P_i \cong R^{i\binom{r}{i+1}}(-i-1)$, and $Q_i \cong R^{\binom{n-r}{i}}(-i)$ and so the twists do not allow the splitting of free addenda. □

By sheafifying the previous resolutions, we get the minimal resolutions of $\mathcal{I}_C$ and $\mathcal{I}_L$ over $\mathcal{O}_{\mathbb{P}^n}$, and of $\mathcal{I}_{C,L}$ over $\mathcal{O}_L$, that we shall use in what follows. As standing notation, the map ε'_i of the resolution of I_C will become $\tilde{\varepsilon}'_i$ after sheafifying the resolution, and the same for the other maps.

As previously explained, to construct a double structure X supported on C we need a surjective morphism $\mu : \frac{\mathcal{I}_C}{\mathcal{I}_C^2} \to \mathcal{L}$ where $\mathcal{L}$ is an invertible sheaf on C. We know that $\mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{I}_C \otimes \mathcal{O}_C$ and so, if we tensorise the resolution of $\mathcal{I}_C$ with $\mathcal{O}_C$ we get $\mathcal{I}_C/\mathcal{I}_C^2 \cong \operatorname{coker}(\tilde{\varepsilon}'_2 \otimes id_{\mathcal{O}_C})$. Moreover, it is easy to prove the following

Proposition 2.4. $\operatorname{coker}(\tilde{\varepsilon}'_2 \otimes id_{\mathcal{O}_C}) \cong \operatorname{coker}(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}) \oplus \mathcal{O}_C^{n-r}(-1)$.

Proof. The restrictions of $\tilde{\varepsilon}'_2 \otimes id_{\mathcal{O}_C}$ to $\tilde{N}_1$ and to $\tilde{Q}_2$ are the null maps because the entries of the mentioned restrictions belong to I_C. □

The curve C is isomorphic to $\mathbb{P}^1$, and an isomorphism $j : \mathbb{P}^1 \to C$ is defined as $j(t : u) = (t^r : t^{r-1}u : \cdots : u^r : 0 : \cdots : 0)$. We have that $j^*(\mathcal{O}_C^{n-r}(-1)) = \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r)$ and $j^*(\mathrm{coker}(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})) = \mathrm{coker}(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})) \cong \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$ (see Lemma 5.4 in [3] for the last isomorphism). Hence, on $\mathbb{P}^1$, the conormal sheaf of $C \subseteq \mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r)$.

Now, we make some effort to explicitly write the previous isomorphism.

Lemma 2.5. *Let $e_1, \ldots, e_r$ and $g_1, \ldots, g_{r-1}$ be the canonical bases of $\mathcal{O}_{\mathbb{P}^1}^r(-r)$ and of $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$, respectively, and let $\psi_r : \wedge^2\mathcal{O}_{\mathbb{P}^1}^r(-r) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$ be defined as*

$$\psi_r(e_p \wedge e_q) = \sum_{h=p}^{q-1} t^{r-h-1}u^{h-1}g_{p+q-1-h}, \qquad \textit{for every } 1 \le p < q \le r.$$

Then, ψ_r is surjective and $\ker(\psi_r) \cong \mathcal{O}_{\mathbb{P}^1}^{\binom{r-1}{2}}(-2r-2)$.

Proof. The map ψ_r is surjective. In fact, $\psi_r(e_1 \wedge e_2), \ldots, \psi_r(e_1 \wedge e_r)$ are linearly independent at each point of $\mathbb{P}^1$ except $(0 : 1)$, while $\psi_r(e_1 \wedge e_r), \ldots, \psi_r(e_{r-1} \wedge e_r)$ are linearly independent at each point of $\mathbb{P}^1$ except $(1 : 0)$. Hence, ψ_r is surjective at every point of $\mathbb{P}^1$ and so it is surjective.

Then, we have the following short exact sequence

$$0 \to \ker(\psi_r) \longrightarrow \wedge^2\mathcal{O}_{\mathbb{P}^1}^r(-r) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \to 0,$$

where $\ker(\psi_r)$ is a locally free $\mathcal{O}_{\mathbb{P}^1}$-module of rank $\binom{r}{2} - (r-1) = \binom{r-1}{2}$. By ([19], Proposition 10.5.1), due to Grothendieck,

$$\ker(\psi_r) \cong \oplus_{i=1}^{\binom{r-1}{2}} \mathcal{O}_{\mathbb{P}^1}(-2r - a_i),$$

for some integers $0 \le a_1 \le \cdots \le a_{\binom{r-1}{2}}$ which are uniquely determined by $\ker \psi_r$.

It is an easy check to prove that

$$u^2 e_p \wedge e_q - tue_p \wedge e_{q+1} - tue_{p+1} \wedge e_q + t^2 e_{p+1} \wedge e_{q+1} \in \ker(\psi_r)$$

for $p = 1, \ldots, r-2$ and $q = p+1, \ldots, r-1$. Of course, if $q = p+1$, the third addendum is missing. Hence, we have $\binom{r-1}{2}$ linearly independent elements of $\ker(\psi_r)$, and so there is a subsheaf of $\ker(\psi_r)$ that is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\binom{r-1}{2}}(-2r-2)$. Hence, the statement holds if we prove that $a_1 = 2$ or equivalently that $\ker \psi_r$ does not contain elements of the form $\sum_{1 \le p < q \le r} l_{pq} e_p \wedge e_q$ with $\deg(l_{pq}) \le 1$. To see this, we proceed by induction on r.

Assume $r = 3$. Then, ψ_3 is represented by the matrix

$$\begin{pmatrix} t & u & 0 \\ 0 & t & u \end{pmatrix}.$$

The syzygy module is then generated by $u^2 e_1 \wedge e_2 - tue_1 \wedge e_3 + t^2 e_2 \wedge e_3$, and so the claim holds if $r = 3$.

Assume the claim holds for ψ_{r-1}. If $\sum_{1\leq p<q\leq r} l_{pq}e_p \wedge e_q \in \ker(\psi_r)$ then, for every $j = 2, \ldots, r$, the l_{pq}'s satisfy the equation

$$l_{1j}t^{r-2} + l_{1,j+1}t^{r-3}u + \cdots + l_{1r}t^{j-2}u^{r-j} + up_j = 0$$

where $p_j \in (l_{23}, \ldots, l_{r-1,r})$, and $p_2 = 0$. Then, $l_{1j} \in (u)$, for every $j = 2, \ldots, r$. Furthermore, for $j = 2$, we have also that $l_{1r} \in (t)$. By degree reasons, $l_{1r} = 0$. By substituting in the equation corresponding to $j = 2$, we get that t can be cancelled and so we get that $l_{1,r-1} \in (t)$. Again by degree reasons, $l_{1,r-1} = 0$. By iterating the argument, we get that $l_{1j} = 0$ for $j = 2, \ldots, r$. The map ψ_r restricted to the span of $e_2 \wedge e_3, \ldots, e_{r-1} \wedge e_r$ is the null map on the first addendum of $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$ and $u\psi_{r-1}$ on the remaining addenda. But then $\sum_{2\leq p<q\leq r} l_{pq}e_p \wedge e_q \in \ker(\psi_r)$ implies $\sum_{2\leq p<q\leq r} l_{pq}e_p \wedge e_q \in \ker(\psi_{r-1})$, and we conclude by induction assumption. □

Theorem 2.6. *With the same hypotheses as Lemma* 2.5*, the sequence*

$$\wedge^3\mathcal{O}_{\mathbb{P}^1}^r(-r) \otimes (\mathcal{O}_{\mathbb{P}^1}^2)^* \xrightarrow{j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})} \wedge^2\mathcal{O}_{\mathbb{P}^1}^r(-r) \xrightarrow{\psi_r} \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \to 0$$

is exact.

Proof. We have only to prove that $\ker(\psi_r) = \mathrm{Im}(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}))$.

To start, we verify that $\psi_r \circ j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}) = 0$. It is a simple computation and its details are

$$\begin{aligned}
&\psi_r(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}))(e_i \wedge e_j \wedge e_h \otimes f_k)) \\
&\quad = t^{r-i+2-k}u^{i-2+k}\psi_r(e_j \wedge e_h) - \\
&\qquad - t^{r-j+2-k}u^{j-2+k}\psi_r(e_i \wedge e_h) + \\
&\qquad + t^{r-h+2-k}u^{h-2+k}\psi_r(e_i \wedge e_j) \\
&\quad = t^{r-i+2-k}u^{i-2+k}(t^{r-h}u^{h-2}g_j + \cdots + t^{r-j-1}u^{j-1}g_{h-1}) \\
&\qquad - t^{r-j+2-k}u^{j-2+k}(t^{r-h}u^{h-2}g_i + \cdots + t^{r-i-1}u^{i-1}g_{h-1}) \\
&\qquad + t^{r-h+2-k}u^{h-2+k}(t^{r-j}u^{j-2}g_i + \cdots + t^{r-i-1}u^{i-1}g_{j-1}) \\
&\quad = 0.
\end{aligned}$$

As last step, we must prove that $\ker(\psi_r) = \mathrm{Im}(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}))$.

In Lemma 2.5, we proved that

$$u^2e_p \wedge e_q - tue_p \wedge e_{q+1} - tue_{p+1} \wedge e_q + t^2e_{p+1} \wedge e_{q+1}$$

for $p = 1, \ldots, r-2$ and $q = p+1, \ldots, r-1$, generate $\ker(\psi_r)$.

Furthermore, we have the equalities

$$\begin{aligned}
&uj^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_p \wedge e_q \wedge e_{q+1} \otimes f_k) - tj^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_{p+1} \wedge e_q \wedge e_{q+1} \otimes f_k) \\
&\quad = t^{r-q+1-k}u^{q-2+k}(u^2e_p \wedge e_q - tue_p \wedge e_{q+1} - tue_{p+1} \wedge e_q + t^2e_{p+1} \wedge e_{q+1})
\end{aligned}$$

for every admissible $p < q$, and so $(\ker(\psi_r))_P = (\mathrm{Im}(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C}))_P)$ at every point $P \in \mathbb{P}^1 \setminus \{(1:0), (0:1)\}$. At $(t:u) = (0:1)$ the equality of the stalks follows from $j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_p \wedge e_q \wedge e_r \otimes f_2) = e_p \wedge e_q$ for every $p = 1, \ldots, r-2, q = p+1, \ldots, r-1$

and the fact that $(\ker(\psi_r))_{(0:1)}$ is generated by $e_p \wedge e_q$ with $p = 1, \dots, r-2, q = p+1, \dots, r-1$. Analogously, we get the claim at $(t:u) = (1:0)$ by computing $j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_1 \wedge e_p \wedge e_q \otimes f_1)$. □

When there is no confusion, we will write ψ instead of ψ_r.

Of course, thanks to the isomorphism j, the map $\mu : \frac{\mathcal{I}_C}{\mathcal{I}_C^2} \to \mathcal{L}$ can be written also as $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ where $j^*(\mathcal{L}) = \mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ for some $a \geq 0$, because the map μ is surjective.

Now, the construction can be rewritten as an algorithm: choose the map μ, and consider the map

$$H^0_*(j_*(\mu \circ (\psi \oplus id))) : H^0_*(C, \wedge^2 \mathcal{O}_C^r(-1) \oplus \mathcal{O}_C^{n-r}(-1)) \longrightarrow H^0_* \mathcal{L}.$$

Let F_1 be a free $H^0_*(C, \mathcal{O}_C)$-module such that the complex

$$F_1 \xrightarrow{\nu} H^0_*(C, \wedge^2 \mathcal{O}_C^r(-1) \oplus \mathcal{O}_C^{n-r}(-1)) \xrightarrow{H^0_*(j_*(\mu \circ (\psi \oplus id)))} H^0_* \mathcal{L}$$

is exact. Let N be a matrix that represents the map ν, and let M be a lifting of N over $R = H^0_*(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$, via the canonical surjective map $R \to R/I_C = H^0_*(C, \mathcal{O}_C)$. The ideal I_X of the doubling X is generated by $I_C^2 + [I_C]M$, where $[I_C]$ is a row matrix with entries equal to the generators of I_C in the same order used to write ψ.

Now, we investigate more deeply the construction. The data we need to construct such a curve X are:

(i) a rational normal curve C of degree r in its linear span L embedded in $\mathbb{P}^n$, for some $n \geq r$, together with an isomorphism $j : \mathbb{P}^1 \to C$;
(ii) a surjective map $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ for some $a \geq 0$.

Remark 2.7. For $r \geq 3$ and $a \geq 0$ there exists always a surjective map μ, while, for $r = n = 2$, there exists a surjective map μ if, and only if, $a = 0$.

Theorem 2.8. *Let X and X' be double structures on two rational normal curves C and C'. Then, $X = X'$ if, and only if, $C = C'$ and the target maps μ and μ' differ by an automorphism of $\mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ after changing j' with j.*

Proof. Assume first that $C = C'$ and $j = j'$. If μ and μ' differ by an automorphism of $\mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ then the maps $\mu \circ \psi$ and $\mu' \circ \psi$ have the same kernel, and so the curves X and X' are defined by the same ideal, i.e., they are equal each other.

Conversely, if X and X' are defined by the same ideal, then X_{red} and X'_{red} are the same curve C, because the supporting curve is defined by the only minimal prime ideal associated to I_X. Up to compose j' with an isomorphism of $\mathbb{P}^1$ we can assume that $j = j'$. The claim follows from [4], (1.1). □

We show with an example how to compute the ideal of such a doubling.

Example. Let $C \subset \mathbb{P}^3 = \mathrm{Proj}(K[x,y,z,w])$ be the twisted cubic curve whose ideal is $I_C = (y^2 - xz, yz - xw, z^2 - yw)$. We want to construct a double structure X

on C contained in $\mathbb{P}^3$. As explained, if we set $a = 1$, we must choose a surjective map

$$\mu : \mathcal{O}_{\mathbb{P}^1}^2(-5) \to \mathcal{O}_{\mathbb{P}^1}(-4).$$

Set $\mu = (t, u)$. The map $\mu \circ \psi$ is given by $\mu \circ \psi = (t^2, 2tu, u^2)$, while the map $H^0_*(j_*(\mu \circ \psi))$ is given by either $(y, 2z, w)$ or $(x, 2y, z)$ (the two apparently different maps agree over $C \setminus \{(1 : 0 : 0 : 0), (0 : 0 : 0 : 1)\}$). Of course, to get the two expressions we multiplied $\mu \circ \psi$ times t and u so that the entries have degree multiple of $r = 3$, and then we used the isomorphism j. The free R/I_C-module F_1 that makes exact the complex

$$F_1 \to (R/I_C)^3(-2) \to H^0_*\mathcal{L}$$

is $F_1 = (R/I_C)^6(-3)$ and a matrix that represents the map $F_1 \to (R/I_C)^3(-2)$ is

$$N = \begin{pmatrix} 2y & 2z & 2w & 0 & 0 & 0 \\ -x & -y & -z & y & z & w \\ 0 & 0 & 0 & -2x & -2y & -2z \end{pmatrix}.$$

By lifting N to a matrix M over R via $R \to R/I_C$ we get $M = N$, where the entries are polynomials in R and no more equivalence classes in R/I_C. The double structure X is then defined by

$$I_X = I_C^2 + [I_C]M.$$

Proposition 2.9. *Let $C \subset \mathbb{P}^r \cong L \subseteq \mathbb{P}^n$ be a rational normal curve of degree r, and let $j : \mathbb{P}^1 \to C$ be an isomorphism. Let $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-r-2+a)$ be a surjective map, and let X be the double structure on C associated to μ. Then, the ideal $I_X = I_C^2 + [I_C]M$, constructed as explained, is saturated.*

Proof. Let $J = I_X^{\mathrm{sat}}$. From the inclusion $J \subseteq I_C$, it follows that there exists a matrix M' such that $J = I_C^2 + [I_C]M'$. Let N, N' be the images of M, M', respectively, when we restrict the last two matrices to R/I_C. The matrices N and N' both present $H^0_*(C, \wedge^2\mathcal{O}_C^r(-1) \oplus \mathcal{O}_C^{n-r}(-1))/\ker(H^0_*(j_*(\mu \circ (\psi \oplus id))))$, and so the columns of N (resp. N') are combination of the ones of N' (resp. N). Hence, $I_X = J$ and I_X is saturated. □

We want to prove some results about families of doublings. Before stating and proving those results, we compute the Hilbert polynomial of a doubling X in terms of the degree of $\mathcal{L}$. Of course, the degree of X is twice the degree of the rational normal curve $C = X_{\mathrm{red}}$ and so we have to compute the genus of X.

Proposition 2.10. *Let X be a doubling of a degree r rational normal curve C defined by a map μ as above. Then, the Hilbert polynomial of X is $P_X(t) = 2rt + a - r$, and so its arithmetic genus g_X is equal to $r + 1 - a$.*

Proof. By construction, the curves C and X and the invertible sheaf $\mathcal{L}$ are related via the short exact sequence (2.2), and so the Hilbert polynomial $P_X(t)$ of X is equal to the sum of the Hilbert polynomial $P_C(t)$ of C and of the Euler character-

istic $\chi\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}^n}(t)$. By restriction to $\mathbb{P}^1$ we get $\chi\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}^n}(t)=\chi\mathcal{O}_{\mathbb{P}^1}(rt-r-2+a)=rt-r-1+a$. The Hilbert polynomial of C is equal to $P_C(t)=rt+1$, and so the claim follows. □

Now, we describe a parameter space for the doublings of the rational normal curves of fixed degree and genus.

From Proposition 2.10, it follows that if we fix degree and genus of X then we fix the degree r of the rational normal curve $C=X_{\mathrm{red}}$ and the twist $a=r+1-g\in\mathbb{Z}$ for the map $\mu:\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)\oplus\mathcal{O}_{\mathbb{P}^1}^{n-r}(-r)\to\mathcal{O}_{\mathbb{P}^1}(-r-2+a)=\mathcal{O}_{\mathbb{P}^1}(-1-g)$.

Let $P(t)=2rt+1-g$ be a polynomial and let $\mathcal{H}ilb_{p(t)}(\mathbb{P}^n)$ be the Hilbert scheme parameterizing locally Cohen-Macaulay curves of $\mathbb{P}^n$ with Hilbert polynomial $P(t)$. Let $H(2r,g,n)$ be the locus in $\mathcal{H}ilb_{p(t)}(\mathbb{P}^n)$ whose closed points correspond to double structures of genus g on smooth rational normal curves of degree r embedded in $\mathbb{P}^n$. Let $H(r,n)$ be the locus in $\mathcal{H}ilb_{rt+1}(\mathbb{P}^n)$ whose closed points are smooth rational normal curves of degree r in $\mathbb{P}^n$. $H(r,n)$ is open in an irreducible component of $\mathcal{H}ilb_{rt+1}(\mathbb{P}^n)$ of dimension $\dim PGL_r-\dim PGL_1+\dim\mathrm{Grass}(n-r,n)=(n+1)(r+1)-4$, where $\mathrm{Grass}(n-r,n)$ is the Grassmannian of the linear spaces of dimension $n-r$ in $\mathbb{P}^n$. Furthermore, there is a natural map $\varphi:H(2r,g,n)\to H(r,n)$ defined as $\varphi(X)=X_{\mathrm{red}}$ where, with abuse of notation, we denote X both the subscheme in $\mathbb{P}^n$ and the closed point in the Hilbert scheme. The fibers of φ are isomorphic to

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)\oplus\mathcal{O}_{\mathbb{P}^1}^{n-r}(-r),\quad\mathcal{O}_{\mathbb{P}^1}(-1-g))/\mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1}(-1-g))$$

and hence they are irreducible and smooth of dimension $(n-1)(r+1-g)+2r-2-n$. Moreover, both $H(2r,g,n)$ and $H(r,n)$ are stable under the action of PGL_n, and so we have proved the following

Theorem 2.11. *$H(2r,g,n)$ is irreducible of dimension*

$$\dim H(2r,g,n)=(n+1)(2r+1-g)-7+2g.$$

Remark 2.12. Of course, we do not know if $H(2r,g,n)$ is an irreducible component of $\mathcal{H}ilb_{P(t)}(\mathbb{P}^n)$. It is reasonable that, under suitable hypotheses on g, it is so. In the last section of the paper, we will study the local properties of $H(4,g,n)$ for $n\geq 3$.

Corollary 2.13. *The dimension of each irreducible component of $\mathcal{H}ilb_{2rt+1-g}(\mathbb{P}^n)$ containing $H(2r,g,n)$ is greater than or equal to $(n+1)(2r+1-g)-7+2g$.*

3. Cohomology estimates

In this section, we show how to compute the Rao function of a double rational normal curve in terms of the data of the construction. To achieve the results, we have to investigate also the curve D defined by the ideal sheaf $\mathcal{I}_C^2$.

At first, we restate a deep result about D due to J. Wahl (see [30], Theorem 2.1), that holds more generally for every Veronese embedding of a projective space.

Theorem 3.1. *Let C be a rational normal curve. Then, $H^1\mathcal{I}_C^2(j)=0$ for $j\neq 2$.*

We want to compute $h^1\mathcal{I}_C^2(2)$ and the ideal $I_D = (I_C^2)^{\text{sat}}$. The following results are probably known in literature, but we add their proofs for completeness. The computation of the generators of I_D rests on some direct calculations and on some basic results on initial ideals. Let us recall the following

Lemma 3.2. *Let I, J be ideals in a polynomial ring R. If $I \subseteq J$ and $in(I) \supseteq in(J)$, then $I = J$, no matter what term ordering we use to compute the initial ideal.*

Proof. If $I \subseteq J$ then $\text{in}(I) \subseteq \text{in}(J)$. From our hypothesis, it follows that $\text{in}(I) = \text{in}(J)$, and the claim follows from ([9], Lemma 15.5). □

Now, we prove some results about the initial ideal of the ideal of a rational normal curve and of its square.

Lemma 3.3. *Let $C \subset \mathbb{P}^r$ be the rational normal curve of degree r generated by the 2×2 minors of the matrix*

$$A = \begin{pmatrix} x_0 & x_1 & \dots & x_{r-1} \\ x_1 & x_2 & \dots & x_r \end{pmatrix}.$$

Then, with respect to the degrevlex ordering of the terms in R, we have that

1. $in(I_C) = \langle x_1, \dots, x_{r-1}\rangle^2$;
2. $in(I_C^2) = \langle x_1, \dots, x_{r-1}\rangle^4 + x_0\langle x_2, \dots, x_{r-1}\rangle^3 + x_r\langle x_2, \dots, x_{r-2}\rangle^3$;
3. $(in(I_C^2))^{\text{sat}} = in(I_C^2) + \langle x_2, \dots, x_{r-2}\rangle^3$.

Proof. Let f_{ij} be the minor given by the ith and jth columns of A, i.e., $f_{ij} = x_{i-1}x_j - x_ix_{j-1}$, with $1 \le i < j \le r$. The leading term of f_{ij}, with respect to the graded reverse lexicographic order, is $\text{in}(f_{ij}) = x_ix_{j-1}$, and so $\text{in}(I_C) \supseteq \langle x_1, \dots, x_{r-1}\rangle^2$. On the other hand, it is easy to verify that $\{f_{12}, \dots, f_{r-1,r}\}$ is a Gröbner basis of I_C and so the first claim holds.

Let $D \subset \mathbb{P}^r$ be the curve defined by the ideal $I_D = H^0_*(\mathcal{I}_C^2) = (I_C^2)^{\text{sat}}$. Thanks to the previous Theorem, the homogeneous parts of the ideals I_C and I_D are related each other from the exact sequence

$$0 \to (I_D)_t \to (I_C)_t \to H^0(\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2+rt)) \to 0$$

for every $t \ge 3$. Hence, $\dim_K(I_D)_t = \binom{r+t}{r} - (r^2t + 2 - r^2)$. It is well known that $\dim_K(I)_t = \dim_K(\text{in}(I))_t$ for every homogeneous ideal ([9], Theorem 15.26). So, $\dim_K(\text{in}(I_C^2))_t \le \binom{r+t}{r} - (r^2t + 2 - r^2)$.

Consider the monomial ideal

$$J = \langle x_1, \dots, x_{r-1}\rangle^4 + x_0\langle x_2, \dots, x_{r-1}\rangle^3 + x_r\langle x_2, \dots, x_{r-2}\rangle^3.$$

We claim that $J = \text{in}(I_C^2)$.

It is a straightforward computation to check that

$$\dim_K(J)_t = \binom{r+t}{r} - (r^2t + 2 - r^2),$$

for $t \ge 4$. For example, it is easy to enumerate the degree t monomials not in J. It follows that $\dim_K(\text{in}(I_C^2))_t \le \dim_K(J)_t$. So, if $\text{in}(I_C^2) \supseteq J$, then $\text{in}(I_C^2) = J$.

Furthermore, we get also that $(I_D)_t = (I_C^2)_t$ for $t \geq 4$, because their homogeneous parts have the same dimension for $t \geq 4$.

Of course, $\mathrm{in}(I_C^2) \supset \langle x_1, \ldots, x_{r-1}\rangle^4$, because of the first claim. Moreover, if $2 \leq i \leq j \leq h \leq r-1$, the leading term of $f_{1i}f_{j,h+1} - f_{1,j+1}f_{i-1,h+1}$ is equal to $x_0x_ix_jx_h$ and so $x_0\langle x_2, \ldots, x_{r-1}\rangle^3 \subset \mathrm{in}(I_C^2)$. Finally, if $2 \leq i \leq j \leq h \leq r-2$, then the leading term of $f_{i,j+1}f_{h+1,r} - f_{i,h+2}f_{j,r} + f_{i-1,r}f_{j+1,h+2}$ is equal to $x_ix_jx_hx_r$ and so $x_r\langle x_2, \ldots, x_{r-2}\rangle^3 \subset \mathrm{in}(I_C^2)$. Because of the previous argument, the second claim follows. The equality in (3) follows from the definition of saturation. □

Now, we can compute the generators of I_D.

Theorem 3.4. *With the same hypotheses as before, let I' be the ideal generated by the 3×3 minors of the matrix*

$$B = \begin{pmatrix} x_0 & x_1 & \ldots & x_{r-2} \\ x_1 & x_2 & \ldots & x_{r-1} \\ x_2 & x_3 & \ldots & x_r \end{pmatrix}.$$

Then, $I_D = I_C^2 + I'$.

Proof. By construction, $\mathrm{in}(I') \supseteq \langle x_2, \ldots, x_{r-2}\rangle^3$, and by Lemma 3.3(3), we have $(\mathrm{in}(I_C^2))^{\mathrm{sat}} = \mathrm{in}(I_C^2) + \langle x_2, \ldots, x_{r-2}\rangle^3$. Moreover, we have the following chain of inclusions

$$\begin{aligned} &\mathrm{in}(I_C^2 + I') \supseteq \mathrm{in}(I_C^2) + \mathrm{in}(I') \supseteq \mathrm{in}(I_C^2) + \langle x_2, \ldots, x_{r-2}\rangle^3 = \\ &= (\mathrm{in}(I_C^2))^{\mathrm{sat}} \supseteq \mathrm{in}((I_C^2)^{\mathrm{sat}}) = \mathrm{in}(I_D). \end{aligned}$$

So, from the inclusion $I_C^2 \subseteq I_D$ and Lemma 3.2, it is enough to show that $I' \subset I_D$.

Let us consider the integers i, j, h with $2 \leq i < j < h \leq r$ and let

$$g_{ijh} = \det \begin{pmatrix} x_{i-2} & x_{j-2} & x_{h-2} \\ x_{i-1} & x_{j-1} & x_{h-1} \\ x_i & x_j & x_h \end{pmatrix}.$$

It is evident that the two following equalities hold

$$g_{ijh} = x_{i-2}f_{jh} - x_{j-2}f_{ih} + x_{h-2}f_{ij} = x_if_{j-1,h-1} - x_jf_{i-1,h-1} + x_hf_{i-1,j-1}.$$

We want to prove that $x_kg_{ijh} \in I_C^2$ for every $k = 0, \ldots, r$.

From Proposition 2.1, we know that

$$x_{i-1}f_{jh} - x_{j-1}f_{ih} + x_{k-1}f_{ij} = x_if_{jh} - x_jf_{ih} + x_hf_{ij} = 0.$$

Hence, the following easy computations prove the claim

$$x_kg_{ijh} = \begin{cases} f_{i-1,k}f_{jh} - f_{j-1,k}f_{ih} + f_{h-1,k}f_{ij} & \text{if } k \geq h \\ f_{i,k+1}f_{j-1,k-1} - f_{j,k+1}f_{i-1,h-1} + f_{k+1,h}f_{i-1,j-1} & \text{if } k < h \end{cases}$$

where, in the last equation, we use the convention that $f_{aa} = 0$ for every a, and $f_{ab} = f_{ba}$ if $a > b$. □

Remark 3.5. If C is a line or a smooth conic in $\mathbb{P}^n$ then I_C^2 is generated by $\binom{r}{2}$ polynomials. By the way, those two cases are the only for which I_C is a complete intersection ideal. If C is a twisted cubic curve, then I_C^2 is saturated. If $r \geq 4$, then I_C^2 is no more saturated, but I_C^2 and its saturation agree from degree 4 on.

Remark 3.6. In the proof of Theorem 2.6, we checked that $\psi(j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_i \wedge e_j \wedge e_h \otimes f_k)) = 0$, and so $j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_i \wedge e_j \wedge e_h \otimes f_k) \in \ker(\mu_| \circ \psi)$ where $\mu_|$ is the restriction of μ to $\wedge^2 \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$. The element $j^*(\tilde{\varepsilon}_2 \otimes id_{\mathcal{O}_C})(e_i \wedge e_j \wedge e_h \otimes f_k)$ corresponds to the minor g_{ijh} and so $(I_C^2)^{\mathrm{sat}} \subseteq I_X$ for every double structure X.

Now, we consider the case $C \subset \mathbb{P}^r \subset \mathbb{P}^n$. As in Section 2, we denote $S = K[x_0, \ldots, x_r]$, and $R = K[x_0, \ldots, x_n]$.

Proposition 3.7. *Let $C \subset L \cong \mathbb{P}^r \subset \mathbb{P}^n$ be a rational normal curve. Let $I_{C,L}, I_C$ be the ideals of C as a subscheme of L and of $\mathbb{P}^n$, respectively, and let I_L be the ideal of L in $\mathbb{P}^n$. Then, $(I_C^2)^{\mathrm{sat}} = ((I_{C,L}^2)^{\mathrm{sat}})^e + (I_{C,L}^e) \cdot I_L + I_L^2 = ((I_{C,L}^2)^{\mathrm{sat}})^e + I_C \cdot I_L$ where the extension is via the natural inclusion $S \hookrightarrow R$.*

Proof. The ideals I_L^2 and $(I_{C,L}^e) \cdot I_L$ are obviously contained in I_C^2 and hence in $(I_C^2)^{\mathrm{sat}}$. Furthermore, to check the inclusion $((I_{C,L}^2)^{\mathrm{sat}})^e \subset (I_C^2)^{\mathrm{sat}}$ it is enough to verify that $f \in (I_C^2)^{\mathrm{sat}}$ for every $f \in ((I_{C,L}^2)^{\mathrm{sat}})^e \cap S$. Let f be a homogeneous polynomial in $((I_{C,L}^2)^{\mathrm{sat}})^e \cap S$. It is easy to check that, for every $i = 0, \ldots, n$, there exists $m_i \in \mathbb{N}$ such that $x_i^{m_i} f \in (I_C^2)$. In fact, if $0 \leq i \leq r$, then there exists $m_i \in \mathbb{N}$ such that $x_i^{m_i} f \in (I_{C,L}^2)^e \cap S \subseteq (I_C^2)$, while, for $r+1 \leq i \leq n$, $x_i^2 f \in I_C^2$ because $I_L^2 \subset I_C^2$. Hence, $((I_{C,L}^2)^{\mathrm{sat}})^e \subseteq (I_C^2)^{\mathrm{sat}}$ and the inclusion $((I_{C,L}^2)^{\mathrm{sat}})^e + (I_{C,L}^e) \cdot I_L + I_L^2 \subseteq (I_C^2)^{\mathrm{sat}}$ follows.

To prove the inverse inclusion, let $f \in (I_C^2)^{\mathrm{sat}}$ be a homogeneous polynomial. There exist $f_1 \in S$ and $f_2 \in I_L$ such that $f = f_1 + f_2$ and the decomposition is unique. For every $i = 0, \ldots, n$, there exists $m_i \in \mathbb{N}$ such that $x_i^{m_i} f \in I_C^2$. Assume $0 \leq i \leq r$. We know that $I_C = I_{C,L}^e + I_L$, and so $I_C^2 = (I_{C,L}^2)^e + I_L \cdot I_C$. Hence, there exist $g_1 \in (I_{C,L}^2)^e \cap S$ and $g_2 \in I_L \cdot I_C$ such that $x_i^{m_i} f = x_i^{m_i} f_1 + x_i^{m_i} f_2 = g_1 + g_2$. It is evident that $x_i^{m_i} f_1 - g_1 = g_2 - x_i^{m_i} f_2 \in S \cap I_L = 0$, and so $x_i^{m_i} f_1 \in (I_{C,L}^2)^e$ and $x_i^{m_i} f_2 \in I_L \cdot I_C$. By definition of saturation, $f_1 \in ((I_{C,L}^2)^e)^{\mathrm{sat}}$. By assumption, $f_2 \in I_L$, and so $f_2 = x_{r+1} f_{2,r+1} + \cdots + x_n f_{2,n}$. Then, $x_i^{m_i} f_{2,j} \in I_C$ for every $j = r+1, \ldots, n$. The ideal I_C is a prime ideal and $x_i^{m_i} \notin I_C$. Hence, $f_{2,j} \in I_C$, $f_2 \in I_L \cdot I_C$ and the proof is complete. □

A consequence of the previous results is that we can compute also $h^1\mathcal{I}_C^2(2)$.

Corollary 3.8. *With the same hypotheses as before, $h^1\mathcal{I}_C^2(2) = \binom{r-1}{2}$.*

Proof. If we tensor the exact sequence

$$0 \to \mathcal{I}_C^2 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r) \to 0$$

by $\mathcal{O}_{\mathbb{P}^n}(2)$ and we take the cohomology, we get

$$h^1\mathcal{I}_C^2(2) = h^0\mathcal{I}_C^2(2) - h^0\mathcal{I}_C(2) + (r-1)h^0\mathcal{O}_{\mathbb{P}^1}(r-2) + (n-r)h^0\mathcal{O}_{\mathbb{P}^1}(r) = \binom{r-1}{2}. \quad \square$$

Thanks to the results on D, we can compute, or at least bound, the Rao function $h^1\mathcal{I}_X(j), j \in \mathbb{Z}$, of X in terms of the map μ which describes the schematic structure of X.

Proposition 3.9. *With the same notation as above, let I_μ be the ideal generated by the entries of μ. Then, it holds*

$$h^1\mathcal{I}_X(j) = \dim_K \left(\frac{K[t,u]}{I_\mu}\right)_{rj-r-2+a} \tag{3.1}$$

for every $j \neq 2$. Moreover,

$$h^1\mathcal{I}_X(2) \leq \dim_K \left(\frac{K[t,u]}{I_\mu}\right)_{r-2+a} + \binom{r-1}{2}.$$

Proof. By construction, we have the short exact sequences

$$0 \to \mathcal{I}_X \to \mathcal{I}_C \to \mathcal{L} \to 0,$$

and

$$0 \to \frac{\mathcal{I}_X}{\mathcal{I}_C^2} \to \frac{\mathcal{I}_C}{\mathcal{I}_C^2} \to \mathcal{L} \to 0$$

where the first map of them both is the inclusion.

The two sequences fit into the larger commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_C^2 & = & \mathcal{I}_C^2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & \mathcal{I}_X & \longrightarrow & \mathcal{I}_C & \longrightarrow & \mathcal{L} & \to 0 & & \\
 & \downarrow & & \downarrow & & \| & & & \\
0 \to & \frac{\mathcal{I}_X}{\mathcal{I}_C^2} & \longrightarrow & \frac{\mathcal{I}_C}{\mathcal{I}_C^2} & \longrightarrow & \mathcal{L} & \to 0. & & \\
 & \downarrow & & \downarrow & & & & & \\
 & 0 & & 0 & & & & &
\end{array}$$

If we twist by $\mathcal{O}_{\mathbb{P}^n}(j)$ and take the cohomology, we get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^0\mathcal{I}_C^2(j) & = & H^0\mathcal{I}_C^2(j) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & H^0\mathcal{I}_X(j) & \longrightarrow & H^0\mathcal{I}_C(j) & \longrightarrow & H^0\mathcal{L}(rj) & \longrightarrow & H^1\mathcal{I}_X(j) & \to 0 \\
 & \downarrow & & \downarrow & & \| & & \downarrow & \\
0 \to & H^0\frac{\mathcal{I}_X}{\mathcal{I}_C^2}(rj) & \longrightarrow & H^0\frac{\mathcal{I}_C}{\mathcal{I}_C^2}(rj) & \longrightarrow & H^0\mathcal{L}(rj) & \longrightarrow & \operatorname{coker}_j & \to 0. \\
 & \downarrow & & \downarrow & & & & & \\
 & H^1\mathcal{I}_C^2(j) & \longrightarrow & H^1\mathcal{I}_C^2(j) & & & & & \\
 & & & \downarrow & & & & & \\
 & & & 0 & & & & &
\end{array}$$

If $j \neq 2$, then $H^1\mathcal{I}_C^2(j) = 0$ and so

$$\operatorname{coker}(H^0\mathcal{I}_X(j) \to H^0\mathcal{I}_C(j)) = \operatorname{coker}(H^0\frac{\mathcal{I}_X}{\mathcal{I}_C^2}(rj) \to H^0\frac{\mathcal{I}_C}{\mathcal{I}_C^2}(rj))$$

as subspaces of $H^0\mathcal{L}(rj)$. Hence, $H^1\mathcal{I}_X(j) \cong \operatorname{coker}_j \cong \left(\frac{K[t,u]}{I_\mu}\right)_{rj-r-2+a}$.

If $j = 2$, we set $\mathcal{A} = \ker(H^0\mathcal{L}(2r) \to H^1\mathcal{I}_X(2))$ and $\mathcal{B} = \ker(H^0\mathcal{L}(2r) \to \operatorname{coker}_2)$. Then, the identity of $H^0\mathcal{L}(2r)$ induces an injective map $\mathcal{A} \to \mathcal{B}$, and the diagram

$$\begin{array}{ccccccc} 0 \to & H^0\mathcal{I}_X(2) & \longrightarrow & H^0\mathcal{I}_C(2) & \longrightarrow & \mathcal{A} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0\frac{\mathcal{I}_X}{\mathcal{I}_C^2}(2r) & \longrightarrow & H^0\frac{\mathcal{I}_C}{\mathcal{I}_C^2}(2r) & \longrightarrow & \mathcal{B} & \to 0 \end{array}$$

induces a surjective map

$$H^1\mathcal{I}_C^2(2) \to \operatorname{coker}(\mathcal{A} \to \mathcal{B}) \cong \ker(H^1\mathcal{I}_X(2) \to \operatorname{coker}_2).$$

Hence, the claim follows from the surjectivity of the map $H^1\mathcal{I}_X(2) \to \operatorname{coker}_2$. □

Example (Example 2 revisited). The genus of the curve X is $g_X = 4 - 1 = 3$, because $a = 1$ (see Proposition 2.10). The map $\mu : \mathcal{O}_{\mathbb{P}^1}^2(-5) \to \mathcal{O}_{\mathbb{P}^1}(-4)$ was defined as $\mu = (t, u)$ and so the Hilbert function of its cokernel is

$$\dim_K \left(\frac{K[t,u]}{(t,u)}\right)_h = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}$$

By Proposition 3.9, the Rao function of X is equal to

$$h^1\mathcal{I}_X(j) = \dim_K \left(\frac{K[t,u]}{(t,u)}\right)_{3j-4} = 0$$

for every $j \neq 2$. To compute $h^1\mathcal{I}_X(2)$ we consider the hyperplane $H = V(w)$ that is general for X, and the exact sequence

$$0 \to \mathcal{I}_X(-1) \to \mathcal{I}_X \to \mathcal{I}_{X\cap H|H} \to 0.$$

If we tensorize by $\mathcal{O}_{\mathbb{P}^3}(2)$ and take the cohomology, we get

$$0 \to H^1\mathcal{I}_X(2) \to H^1\mathcal{I}_{X\cap H|H}(2) \to 0,$$

because $H^2\mathcal{I}_X(1) = 0$. It is easy to verify that $h^1\mathcal{I}_{X\cap H|H}(2) = h^0\mathcal{I}_{X\cap H|H}(2)$ and so $h^1\mathcal{I}_X(2) = 0$ if, and only if, $X \cap H$ is not contained in any conic of H. But $\left(\frac{I_X + wR}{wR}\right)^{\text{sat}} = \langle 2y^3 - 3xyz, y^2z - 2xz^2, yz^2, z^3\rangle$ and so X is an arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$.

4. Arithmetically Gorenstein double rational normal curves

In this section, we want to describe the arithmetically Gorenstein curves among the double structures on rational normal curves. At first, we characterize the possible triples $(2r, g, n)$ and then we study the possible cases one at a time.

To start, we recall the definition of arithmetically Gorenstein curve.

Definition 4.1. A curve $X \subset \mathbb{P}^n$ is arithmetically Gorenstein if its homogeneous coordinate ring R_X is a Gorenstein ring, or, equivalently, if R_X is Cohen-Macaulay and its canonical sheaf ω_X is a twist of the structure sheaf.

Now, we look for triples $(2r, g, n)$ for which the property of being arithmetically Gorenstein is allowed.

Proposition 4.2. *Let $C \subset L \cong \mathbb{P}^r \subseteq \mathbb{P}^n, n \geq 3$, be a rational normal curve of degree r, let $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-1-g)$ be a surjective map, and let X be the double structure on C defined by μ. If X is a non-degenerate arithmetically Gorenstein curve, then either $(2r, g, n) = (2r, r+1, r)$ or $(2r, g, n) = (2r, 1, 2r-1)$.*

Proof. If X is an arithmetically Gorenstein curve, then the second difference $\Delta^2 h_X$ of its Hilbert function h_X is a symmetric function. Moreover, if X is non-degenerate, then $\Delta^2 h_X(1) = n - 1$. We have the equality $2r = \deg(X) = \sum_{j=0}^{\infty} \Delta^2 h_X(j)$ and so we get $r \leq n \leq 2r$, where the first inequality comes from the general setting, and the second one from $\Delta^2 h_X(0) = 1$. If X is an arithmetically Gorenstein curve, then $\Delta^2 h_X$ is the Hilbert function of the Artinian ring $R/\langle I_X, h_1, h_2 \rangle$ where h_1, h_2 are two linear forms, general with respect to X. In particular, if $\Delta^2 h_X(j) = 0$ for some $j > 0$, then $\Delta^2 h_X(k) = 0$ for every $k \geq j$. From the above discussion and inequalities, we get that there are either 3 or 4 non-zero entries in $\Delta^2 h_X$. In the first case, then $\Delta^2 h_X = (1, 2r-2, 1)$ and $(2r, g, n) = (2r, 1, 2r-1)$. In the second case, then $\Delta^2 h_X = (1, r-1, r-1, 1)$ and $(2r, g, n) = (2r, r+1, r)$. □

Remark 4.3. If X is a double conic in $\mathbb{P}^2$, then $I_X = \langle q^2 \rangle$ where $q = xz - y^2$ defines the smooth conic that supports X. X is arithmetically Gorenstein with $\omega_X = \mathcal{O}_X(1)$.

Now, we characterize the arithmetically Gorenstein double curves among the ones we can construct with given triple $(2r, g, n)$.

Theorem 4.4. *Let $(2r, g, n) = (2r, r+1, r)$. For every non-zero map $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \to \mathcal{O}_{\mathbb{P}^1}(-r-2)$ we get a non-degenerate arithmetically Cohen-Macaulay curve X. Furthermore, if μ is general, then X is arithmetically Gorenstein. If $\mu = (\alpha^{r-2}, \alpha^{r-3}\beta, \ldots, \beta^{r-2})$ then X is contained in a cone over a rational normal curve $C' \subset \mathbb{P}^{r-1}$.*

Proof. Let $\mu = (a_0, \ldots, a_{r-2}) \neq 0$ with $a_i \in K$. Let X be the curve we get by doubling a rational normal curve C via μ.

We prove that X is an arithmetically Cohen-Macaulay curve. By Proposition 3.9, the surjectivity of μ implies that $h^1\mathcal{I}_X(j) = 0$ for $j \neq 2$, and so we have to prove

that $h^1\mathcal{I}_X(2)=0$, too. The map $\mu\circ\psi$ can be written also as the composition $\mu'\circ\psi'$ where $\mu':\mathcal{O}_{\mathbb{P}^1}^{r-1}(-2r)\to\mathcal{O}_{\mathbb{P}^1}(-r-2)$ is defined as $\mu'=(t^{r-2},t^{r-3}u,\ldots,u^{r-2})$ and $\psi':\wedge^2\mathcal{O}_{\mathbb{P}^1}^r(-r)\to\mathcal{O}_{\mathbb{P}^1}^{r-1}(-2r)$ is defined by the matrix obtained from the one of ψ by substituting $t^iu^j, i+j=r-2$, with a_j, where ψ was defined in Theorem 2.6. The matrix of ψ' is full rank for whatever non-zero map μ, and so it has exactly $\binom{r}{2}-(r-1)=\binom{r-1}{2}$ linearly independent degree 0 syzygies. Furthermore, a degree 0 syzygy of $\mu\circ\psi$ is a degree 0 syzygy of ψ' and hence, $h^0\mathcal{I}_X(2)=\binom{r-1}{2}$. From the exact sequence

$$0\to H^0\mathcal{I}_X(2)\to H^0\mathcal{I}_C(2)\to H^0\frac{\mathcal{I}_C}{\mathcal{I}_X}(2)\to H^1\mathcal{I}_X(2)\to 0,$$

and from $h^0\mathcal{I}_C(2)=\binom{r}{2}, h^0\frac{\mathcal{I}_C}{\mathcal{I}_X}(2)=h^0\mathcal{O}_{\mathbb{P}^1}(r-2)=r-1, h^0\mathcal{I}_X(2)=\binom{r-1}{2}$ we get that $h^1\mathcal{I}_X(2)=0$.

The curve C is rational normal and so $\mathrm{Pic}(C)=\mathbb{Z}$. We know that the line bundle $\mathcal{L}$ verifies $j^*(\mathcal{L})=\mathcal{O}_{\mathbb{P}^1}(-r-2)$ and so $\mathcal{L}=\omega_C(-1)$ where ω_C is the canonical sheaf of C. Hence, the curve X is defined via the exact sequence

$$0\to\mathcal{I}_X\to\mathcal{I}_C\to\omega_C(-1)\to 0.$$

From the exact sequence, we can compute the Hilbert function of X and its second difference. In particular, we get $\Delta^2h_X=(1,r-1,r-1,1)$, as expected.

Assume now that $\mu=(\alpha^{r-2},\alpha^{r-3}\beta,\ldots,\beta^{r-2})$ for some $(\alpha,\beta)\in K^2\backslash\{(0,0)\}$. The 2×2 minors of the matrix

$$\begin{pmatrix}\beta x_0-\alpha x_1 & \beta x_1-\alpha x_2 & \ldots & \beta x_{r-2}-\alpha x_{r-1}\\ \beta x_1-\alpha x_2 & \beta x_2-\alpha x_3 & \ldots & \beta x_{r-1}-\alpha x_r\end{pmatrix}$$

define a cone in $\mathbb{P}^r$ over a rational normal curve of $\mathbb{P}^{r-1}$. We want to prove that they belong to I_X. To this end, let $1\le i<j\le r-1$. The minor $F_{i,j}$ given by the ith and jth columns is equal to

$$\begin{aligned}F_{i,j}&=\det\begin{pmatrix}\beta x_{i-1}-\alpha x_i & \beta x_{j-1}-\alpha x_j\\ \beta x_i-\alpha x_{i+1} & \beta x_j-\alpha x_{j+1}\end{pmatrix}\\&=\beta^2f_{ij}-\alpha\beta f_{i,j+1}-\alpha\beta f_{i+1,j}+\alpha^2f_{i+1,j+1}\end{aligned}$$

where $f_{pq}=x_{p-1}x_q-x_px_{q-1}$ is a generator of I_C. The claim follows if we prove that

$$\beta^2e_i\wedge e_j-\alpha\beta e_i\wedge e_{j+1}-\alpha\beta e_{i+1}\wedge e_j+\alpha^2e_{i+1}\wedge e_{j+1}$$

is a syzygy of $\mu\circ\psi$ for every $1\le i<j\le r-2$. Since $\mu(g_h)=\alpha^{r-h-1}\beta^{h-1}, h=1,\ldots,r-1$, where $g_1,\ldots,g_{r-1}$ is the canonical basis of $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$, then, we have

$$\begin{aligned}&\mu\circ\psi(\beta^2e_i\wedge e_j-\alpha\beta e_i\wedge e_{j+1}-\alpha\beta e_{i+1}\wedge e_j+\alpha^2e_{i+1}\wedge e_{j+1})\\&=\mu(t^{r-j-1}u^{j-1}(-\alpha\beta g_i+\alpha^2g_{i+1})+t^{r-j}u^{j-2}(\beta^2g_i-2\alpha\beta g_{i+1}+\alpha^2g_{i+2})+\ldots\\&\quad\cdots+t^{r-i-2}u^i(\beta^2g_{j-2}-2\alpha\beta g_{j-1}+\alpha^2g_j)+t^{r-i-1}u^{i-1}(\beta^2g_{j-1}-\alpha\beta g_j))=0\end{aligned}$$

and the claim follows.

Let $(\alpha, \beta) = (0, 1)$. Then, as explained before, the double structure X associated to $\mu = (0, \dots, 0, 1)$ is contained in the cone over the rational normal curve C' of $\mathbb{P}^{r-1} \cong H = V(x_r)$ defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{r-2} \\ x_1 & x_2 & \dots & x_{r-1} \end{pmatrix}.$$

The hyperplane H intersects C in $A(1 : 0 : \cdots : 0)$ with multiplicity r, and so $X \cap H$ is supported on A and has degree $2r$. Moreover, $X \cap H$ is contained in the rational normal curve C' and so it is arithmetically Gorenstein. In fact, $\deg(X \cap H) = 2\deg(C') + 2$, $Pic(C') \cong \mathbb{Z}$, and so $X \cap H \in |2H' - K'|$, where H' is the class of a hyperplane section of C' and K' is the canonical divisor of C', and every divisor of the system $dH' - K'$ is arithmetically Gorenstein, for every $d \geq 0$ ([23], Theorem 4.2.8). Hence, X is arithmetically Cohen-Macaulay with an arithmetically Gorenstein hyperplane section, i.e., X is arithmetically Gorenstein, because the graded Betti numbers of the minimal free resolutions of I_X and $I_{X \cap H|H}$ are the same. In fact, the only irreducible component of X is C that is non-degenerate, and so x_r is not a 0-divisor for R/I_X. Hence, the hyperplane $H = V(x_r)$ we considered is general enough for X to let the proof of ([23], Theorem 1.3.6) work (see also [23], Remark 1.3.9).

The family $H(2r, r + 1, r)$ is irreducible, and the arithmetically Gorenstein locus in it is not empty. From the semicontinuity of the Betti numbers in an irreducible family ([7]) it follows that an open subscheme of $H(2r, r + 1, r)$ parameterizes arithmetically Gorenstein schemes and the claim follows. □

Remark 4.5. In the case we just studied, the degree and the genus of X are the ones of a canonical curve in $\mathbb{P}^r$, and so the result we proved is not unexpected. In fact, in [12], the author proved that $H(2r, r + 1, r)$ is contained in the closure of the component of the canonical curves.

Now, we consider the second case, namely $(2r, g, n) = (2r, 1, 2r - 1)$.

Theorem 4.6. *Let $(2r, g, n) = (2r, 1, 2r-1)$. Then for a general map $\mu : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-2)$ we get a non-degenerate arithmetically Gorenstein curve, where general means that $\mu_| : \mathcal{O}_{\mathbb{P}^1}^{r-1}(-r) \to \mathcal{O}_{\mathbb{P}^1}(-2)$ has no degree 0 syzygy.*

Proof. Assume that the restriction μ_1 of μ to $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r-2)$ is the null map.

If $r = 2$, then $\mu = (0, 1)$ and hence X is defined by the ideal $I_X = (x_0x_2 - x_1^2, x_3^2)$. Then X is a complete intersection of a cone and a double plane.

Assume now that $r \geq 3$, and furthermore assume that the restriction μ_2 of μ to $\mathcal{O}_{\mathbb{P}^1}^{r-1}(-r)$ is given by $\mu_2 = (t^{r-2}, t^{r-3}u, \dots, u^{r-2})$. Of course, μ_2 has no degree 0 syzygy. The map $\mu \circ \psi$ is given by the matrix $(0, \dots, 0, t^{r-2}, t^{r-3}u, \dots, u^{r-2})$. By using the procedure described in Section 2, we get that the double structure X is defined by the ideal $I_X = I_{C,L} + I_L^2 + J$ where $J = \langle x_1x_{r+1} - x_0x_{r+2}, \dots, x_rx_{r+1} - x_{r-1}x_{r+2}, \dots, x_1x_{2r-2} - x_0x_{2r-1}, \dots, x_rx_{2r-2} - x_{r-1}x_{2r-1} \rangle$. It is evident that the

ideal I_S defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & \dots & x_{r-1} & x_{r+1} & \dots & x_{2r-2} \\ x_1 & \dots & x_r & x_{r+2} & \dots & x_{2r-1} \end{pmatrix}$$

is contained in I_X, i.e., X is contained in S which is a smooth rational normal scroll surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ embedded via the complete linear system $|\xi + rf|$ ([9], exercise A2.22), where ξ is the class of the rational normal curve of minimal degree $r-2$ contained in S and f is a fibre. On S, we have that $\xi^2 = -2, \xi \cdot f = 1, f^2 = 0$. Moreover, the canonical divisor of S is $K_S = -2\xi - 2f$, and the hyperplane section class is $H = \xi + rf$ ([17], Lemma 2.10). Then $X \in |a\xi + bf|$, with $a = 2, b = 4$, by adjunction, and so X is an anticanonical divisor on S and so it is a non-degenerate arithmetically Gorenstein curve ([23], Theorem 4.2.8). To complete the proof, we show that the curve X we constructed before is the only double structure on C of arithmetic genus 1, up to automorphisms of $\mathbb{P}^{2r-1}$, which is the content of next Theorem 4.7. □

Theorem 4.7. *Let $C \subset L \cong \mathbb{P}^r \subset \mathbb{P}^{2r-1}$ be a rational normal curve of degree r. Then, there exists only one non-degenerate double structure X on C of arithmetic genus 1, up to automorphisms of $\mathbb{P}^{2r-1}$.*

Proof. To make the proof more readable, we choose the coordinates of $\mathbb{P}^{2r-1}$ as $x_0, \dots, x_r, y_1, \dots, y_{r-1}$, where $L = V(y_1, \dots, y_{r-1})$.

As in the proof of the previous theorem, let μ_1 and μ_2 be the restrictions of μ to $\mathcal{O}^{r-1}_{\mathbb{P}^1}(-r-2)$ and to $\mathcal{O}^{r-1}_{\mathbb{P}^1}(-r)$, respectively. Assume first that $\mu_2 = (l_1, \dots, l_{r-1})$. The forms $l_1, \dots, l_{r-1}$ are linearly dependent if, and only if, they have a degree 0 syzygy, that, of course, is also a degree 0 syzygy of $\mu \circ \psi$. So, X is degenerate if, and only if, $l_1, \dots, l_{r-1}$ are linearly dependent. Hence, we can assume that $l_1, \dots, l_{r-1}$ are linearly independent, and so there exists an invertible matrix $P \in GL_{r-1}(K)$ such that

$$\begin{pmatrix} l_1 \\ \vdots \\ l_{r-1} \end{pmatrix} = P \begin{pmatrix} t^{r-2} \\ \vdots \\ u^{r-2} \end{pmatrix}.$$

Going back to the construction, it is clear that the choice of the generators of L plays no role when we restrict the maps to $\mathbb{P}^1$ and so, if we say that I_L is generated by

$$P^{-1} \begin{pmatrix} y_1 \\ \vdots \\ y_{r-1} \end{pmatrix}$$

and we change bases in $\mathcal{O}^{r-1}_{\mathbb{P}^1}(-r)$ by using P^{-1} we get that I_L is generated by $y_1, \dots, y_{r-1}$ and $\mu_2 = (t^{r-2}, \dots, u^{r-2})$.

Let $\mu_1 = (p_1, \dots, p_{r-1})$, where $p_i = p_{i0}t^r + p_{i1}t^{r-1}u + \dots + p_{ir}u^r$. The map $\mu \circ \psi$ has the first $\binom{r}{2}$ entries that are combinations of $t^{r-2}, \dots, u^{r-2}$ with coefficients $p_1, \dots, p_{r-1}$ and the last $r-1$ entries which are equal to $t^{r-2}, \dots u^{r-2}$. Hence, the

syzygies of $\mu\circ\psi$ can be easily computed and we get that the defining ideal I_X of X is generated by $x_{i-1}x_{i+1}-x_i^2-p_iy_i, i=1,\dots,r-1$, by $x_ix_{j+1}-x_{i+1}x_j-y_{i+1}p_j-\dots-y_jp_{i+1}, 0\le i<j-1\le r-2$, by $x_iy_j-x_{i-1}y_{j+1}, i=1,\dots,r, j=1,\dots,r-2$, and by $y_iy_j, 1\le i\le j\le r-1$, where, with abuse of notation, we set p_i also the only linear form in $x_0,\dots,x_r$ that is equal to p_i when restricted to $\mathbb{P}^1$, i.e., $p_i=p_{i0}x_0+\dots+p_{ir}x_r$.

We look for the required change of coordinates in the form

$$x_i=z_i+a_{i1}y_1+\dots a_{i,r-1}y_{r-1}\quad i=0,\dots,r$$

and we fix the remaining variables $y_1,\dots,y_{r-1}$.

Our goal is to prove that we can choose the a_{ij}'s in such a way that, in the new coordinate system, X is defined by the ideal J generated by

$$z_{i-1}z_{i+1}-z_i^2, z_iz_{j+1}-z_{i+1}z_j, z_iy_j-z_{i-1}y_{j+1}, y_iy_j,$$

where the indices vary in the same ranges as before.

If we apply the change of coordinates to the last generators of I_X then they do not change, because the variables $y_i,\dots y_{r-1}$ are fixed. If we apply the change of coordinates to the generators of the form $x_iy_j-x_{i-1}y_{j+1}$, we get that $z_iy_j-z_{i-1}y_{j+1}\in J$ because $y_hy_k\in J$, for $i=1,\dots,r, j=1,\dots,r-2$, and $1\le h\le k\le r-1$.

By applying the change of coordinates to $x_{i-1}x_{i+1}-x_i^2-y_ip_i$ we get

$$\begin{aligned}
&\left(z_{i-1}+\sum_{j=1}^{r-1}a_{i-1,j}y_j\right)\left(z_{i+1}+\sum_{j=1}^{r-1}a_{i+1,j}y_j\right)-\left(z_i+\sum_{j=1}^{r-1}a_{ij}y_j\right)^2-y_i\sum_{k=0}^{r}p_{ik}z_k\\
&\quad=z_{i-1}z_{i+1}-z_i^2-\sum_{j=1}^{r-1}a_{i-1,j}z_{i+1}y_j-\sum_{j=1}^{r-1}a_{i+1,j}z_{i-1}y_j+2\sum_{j=1}^{r-1}a_{ij}z_iy_j\\
&\qquad-\sum_{k=0}^{r}p_{ik}z_ky_i=^{(*)}z_{i-1}z_{i+1}-z_i^2-\sum_{j=1}^{r-1}a_{i-1,j}z_{i+j+2-r}y_{r-1}\\
&\qquad-\sum_{j=1}^{r-1}a_{i+1,j}z_{i+j-r}y_{r-1}+2\sum_{j=1}^{r-1}a_{ij}z_{i+j+1-r}y_{r-1}-\sum_{k=0}^{r}p_{ik}z_{k+i+1-r}y_{r-1},
\end{aligned}$$

where, in $(*)$, we use the fact that $z_iy_j-z_{i-1}y_{j+1}\in J$ and the convention that we can use z_xy_{r-1} with $x<0$ to mean z_0y_{r+x-1}.

Hence, we get the following linear equations in the a_{ij}'s:

$$a_{i-1,h-1}-2a_{ih}+a_{i+1,h+1}-p_{ih}=0 \tag{4.1}$$

for $h=0,\dots,r$ where we assume that $a_{ij}=0$ if $j\le 0$ or $j\ge r$, for whatever i.

With analogous computations, if we apply the change of coordinates to $x_i x_{j+1} - x_{i+1}x_j - y_{i+1}p_j - \cdots - y_j p_{i+1}$ we get the following linear equations

$$a_{i,m-j-2+r} + a_{j+1,m-i-1+r} - a_{j,m-i-2+r} - a_{i+1,m-j-1+r} - \sum_{t=i+1}^{j} p_{i+j+1-t,m-t-1+r} = 0.$$

It is an easy computation to show that those last equations depend linearly from the previous ones. For example, if we subtract from the last equation the one among (4.1) we get setting $i = j, h = m - i - 2 + r$, we have the relation

$$a_{i,m-j-2+r} + a_{j,m-i-2+r} - a_{j-1,m-i-3+r} - a_{i+1,m-j-1+r} - \sum_{t=i+2}^{j} p_{i+j+1-t,m-t-1+r} = 0,$$

which is again of the same form, but with smaller difference between the first subscripts. By iterating, we get that all of them linearly depend from the equations (4.1).

Now, we prove that the linear system (4.1) has one solution. To this aim, we collect the equation according to the difference $i - h$ of the subscripts of the variables involved. In fact, notice that in each equation, the difference is constant.

At first, assume that the difference is $i - h = 0$. Then, we get the following linear system

$$\begin{cases} -2a_{11} + a_{22} = p_{11} \\ a_{11} - 2a_{22} + a_{33} = p_{22} \\ \vdots \\ a_{i-1,1-i} - 2a_{ii} + a_{i+1,i+1} = p_{ii} \\ \vdots \\ a_{r-2,r-2} - 2a_{r-1,r-1} = p_{r-1,r-1} \end{cases}$$

The coefficient matrix $M_{r-1} = (m_{ij})$ has entries equal to

$$m_{ij} = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and its determinant is equal to $2p + 1$ if $r - 1 = 2p$, or to $-2p$ if $r - 1 = 2p - 1$. In fact, by the Laplace formula,

$$\det(M_{r-1}) = -2\det(M_{r-2}) - \det(M_{r-3}),$$

by direct computation $\det(M_1) = -2, \det(M_2) = 3$ and the claim can be easily proved by induction. Hence, the previous linear system has one solution, by Cramer's rule.

Assume now that the difference is equal to $i-h=k>0$. Hence, the corresponding linear system is

$$\begin{cases} a_{k+1,1} = p_{k0} \\ -2a_{k+1,1}+a_{k+2,2} = p_{k+1,1} \\ a_{k+1,1}-2a_{k+2,2}+a_{k+3,3} = p_{k+2,2} \\ \vdots \\ a_{r-2,r-2-k}-2a_{r-1,r-1-k}+a_{r,r-k} = p_{r-2,r-2-k} \end{cases}$$

and it has one solution for every k.

Analogously, the system with $i-h=k<0$ has one solution for every k and the claim follows. □

Remark 4.8. Let X be the double structure on a rational normal curve defined in the proof of Theorem 4.6. There is a natural map $\Psi : Aut(\mathbb{P}^{2r-1}) \to H(2r,1,2r-1)$ defined as $\Psi(g)=g(X)$ where $g(X)$ is the double structure we get by applying g to X. Previous Theorem 4.7 is equivalent to $\ker\Psi = Aut(C)$. In fact, every automorphism g of C extends to an automorphism g' of L that fixes C. g' can be further extended to an automorphism g'' of $\mathbb{P}^{2r-1}$ that fixes L. For such a g'' we have that $g''(X)=X$. Hence, $Aut(C)\subseteq\ker\Psi$. By a dimension count, we get that Ψ is surjective if, and only if, $\ker\Psi = Aut(C)$.

Now, we apply the previous results to Gorenstein liaison.

Corollary 4.9. *A rational normal curve $C\subset\mathbb{P}^r\cong L\subseteq\mathbb{P}^n$ of degree r is self-linked if, and only if, either $n=r$ or $n=2r-1$.*

Proof. C is self-linked if, and only if, there exists a double structure supported on C that is arithmetically Gorenstein. The claim is then a direct consequence of Theorems 4.4 and 4.6. □

5. Double conics

In this section, we prove that the general double structure of genus $g\leq -2$ supported on a smooth conic is a smooth point in its Hilbert scheme. Moreover, they are the general element of an irreducible component in the same range of the arithmetic genus. On the other hand, if such double structures are contained in $\mathbb{P}^3$, and their genus satisfies $g\geq -1$, then we identify the general element of the irreducible component containing the considered double structures. To achieve the result, we compare the dimension of the family of the double structures of fixed genus with the dimension of $H^0(X,\mathcal{N}_X)$, global sections of the normal sheaf of a suitable double conic X. In fact, it is well known that $H^0(X,\mathcal{N}_X)$ can be identified with the tangent space to the Hilbert scheme at X. To get the desired results, we consider first a suitable double conic $X\subset\mathbb{P}^3$. We describe its ideal I_X and the minimal free resolution

$$0\to F_3\to F_2\to F_1\to I_X\to 0,$$

where F_i is a free $R = K[x, y, z, w]$-module, then the R-module structure of the global sections of $H^0_*(X, \mathcal{O}_X) = \bigoplus_{j\in\mathbb{Z}} H^0(X, \mathcal{O}_X(j))$ and finally we compute $H^0(X, \mathcal{N}_X)$ as the degree 0 elements of

$$\ker(\mathrm{Hom}(F_1, H^0_*(X, \mathcal{O}_X)) \to \mathrm{Hom}(F_2, H^0_*(X, \mathcal{O}_X))).$$

The result for a general double conic $X \subset \mathbb{P}^n$ follows from the smoothness of the Hilbert scheme at a degenerate double conic. Furthermore, in $\mathbb{P}^3$, we distinguish the case $g(X)$ odd from the case $g(X)$ even, because the ideals have a different minimal number of generators, and so their minimal free resolutions have not comparable free modules and maps. Of course, even if there are differences, we use the same arguments in both cases.

5.1. Case g odd, i.e., $a = 2b$

Throughout this subsection, we will use the following running notation. We set $R = K[x, y, z, w]$, and $C \subset \mathbb{P}^3 = \mathrm{Proj}(R)$ is the conic defined by the ideal $I_C = (xz - y^2, w)$. Let $j : \mathbb{P}^1 \to C$ be the isomorphism defined as $j(t : u) = (t^2 : tu : u^2 : 0)$. Finally, we set $\mu : \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-4 + 2b)$ to be the map defined as $\mu = (u^{2b}, t^{2b-2})$.

Proposition 5.1. *If X is the doubling of C associated to μ, then*

1. $I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, x^{b-1}(xz - y^2) - z^b w\rangle$;
2. $w^2, w(xz - y^2), (xz - y^2)^2, x^{b-1}(xz - y^2) - z^b w$ *is a Gröbner basis of I_X with respect to the reverse lexicographic order;*
3. *the minimal free resolution of I_X is*

$$0 \to R(-b-4) \xrightarrow{\delta_2} \begin{matrix} R(-4) \\ \oplus \\ R(-5) \\ \oplus \\ R(-b-2) \\ \oplus \\ R(-b-3) \end{matrix} \xrightarrow{\delta_1} \begin{matrix} R(-2) \\ \oplus \\ R(-3) \\ \oplus \\ R(-4) \\ \oplus \\ R(-b-1) \end{matrix} \longrightarrow I_X \to 0,$$

where the maps δ_1 and δ_2 will be described in the proof.

Proof. The syzygies of $H^0_*(j_*\mu)$ are generated by $M = \begin{pmatrix} x^{b-1} \\ -z^b \end{pmatrix}$ and so the saturated ideal I_X of X is generated by $I_C^2 + [I_C]M$, that is to say,

$$I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, x^{b-1}(xz - y^2) - z^b w\rangle.$$

The generators are a Gröbner basis of I_X because their S-polynomials reduces to 0 via themselves ([9], Theorem 15.8). Moreover, the free R-module F_1 follows. Let $(g_1, \ldots, g_4)$ be a syzygy of I_X. Then, in R, we have

$$w^2 g_1 + w(xz - y^2)g_2 + (xz - y^2)^2 g_3 + x^{b-1}(xz - y^2)g_4 - z^b w g_4 = 0,$$

that can be rewritten as $w(wg_1 + (xz - y^2)g_2 - z^b g_4) + (xz - y^2)((xz - y^2)g_3 + x^{b-1}g_4) = 0$. The two polynomials $w, xz - y^2$ are a regular sequence, and so there exists $g \in R$ such that

$$\begin{aligned} wg + (xz - y^2)g_3 + x^{b-1}g_4 &= 0 \\ wg_1 + (xz - y^2)(g_2 - g) - z^b g_4 &= 0. \end{aligned}$$

Both $w, xz - y^2, x^{b-1}$ and $w, xz - y^2, z^b$ form a regular sequence, and so we have

$$\begin{pmatrix} g \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0 & x^{b-1} & -(xz - y^2) \\ -x^{b-1} & 0 & w \\ xz - y^2 & -w & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

and

$$\begin{pmatrix} g_1 \\ g_2 - g \\ -g_4 \end{pmatrix} = \begin{pmatrix} 0 & z^b & -(xz - y^2) \\ -z^b & 0 & w \\ xz - y^2 & -w & 0 \end{pmatrix} \begin{pmatrix} f_4 \\ f_5 \\ f_6 \end{pmatrix}.$$

By comparing the value of g_4 from the two expressions above, we get the equation $(xz - y^2)(f_1 + f_4) = w(f_2 + f_5)$. By using the same argument as before, there exists $h \in R$ such that $f_4 = -f_1 + wh, f_5 = -f_2 + (xz - y^2)h$. Hence, it holds

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} -(xz - y^2) & 0 & -z^b & 0 \\ w & -(xz - y^2) & x^{b-1} & z^b \\ 0 & w & 0 & -x^{b-1} \\ 0 & 0 & -w & xz - y^2 \end{pmatrix} \begin{pmatrix} f_6 - z^b h \\ f_3 \\ f_2 \\ f_1 \end{pmatrix},$$

and the 4×4 matrix represents the map δ_1. Of course, the free R-module F_2 follows from F_1 and from the degrees of the entries of the map δ_1.

The second syzygies of I_X can be computed as the first ones, and we get

$$\delta_2 = \begin{pmatrix} -z^b \\ x^{b-1} \\ xz - y^2 \\ -w \end{pmatrix}.$$

□

The last statement of next Proposition is due to the anonymous referee that we thank once more.

Proposition 5.2. *X has genus $g(X) = 3 - 2b$, and the Hartshorne-Rao function of X is*

$$h^1\mathcal{I}_X(j) = \begin{cases} 2(j+b) - 3 & \text{if } -b+2 \le j \le 0 \\ 2b - 2 & \text{if } j = 1 \\ 2(b-j) + 1 & \text{if } 2 \le j \le b \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the Hartshorne-Rao module of X is isomorphic to

$$R/\langle w, xz - y^2, x^{b-1}, z^b \rangle(-b+2).$$

Proof. The genus and the Hartshorne-Rao function can be computed by using results from Section 3.

For the last statement, we first remark that double conics are minimal curves in the sense of [22]. Otherwise, a double conic would be bilinked down to a degree two curve. But the Hartshorne-Rao function of a degree two curve increases by at most one, and so we can exclude the case. For minimal curves, the map $\delta_2^\vee$ begins a minimal free resolution of $H^1_*\mathcal{I}_X$ and so the claim follows because the entries of δ_2 are a regular sequence. □

Now, we can compute the dimension of the degree d global sections of the structure sheaf of X. In fact, it holds

Proposition 5.3. *$h^0(X, \mathcal{O}_X(d)) = 4d + 2b - 2$ if $d \geq 2$.*

Proof. The short exact sequence

$$0 \to \frac{R}{I_X} \to H^0_*(X, \mathcal{O}_X) \to H^1_*\mathcal{I}_X \to 0$$

allows us to prove the result. □

Now, we describe the elements of $H^0(X, \mathcal{O}_X(d))$ for every $d \geq 2$. To start, we can easily describe the elements of $H^0(X, \mathcal{O}_X(d))$ for $d \geq b+1$, because, in the considered range, we have $(R/I_X)_d = H^0(X, \mathcal{O}_X(d))$, and so it holds

Proposition 5.4. *Let $d \geq b+1$. Then, $H^0(X, \mathcal{O}_X(d)) = V_d$ where*

$$V_d = \{p_1 + yp_2 + (xz - y^2)p_3 + (xz - y^2)yp_4 + wx^{d-b}p_5 + wx^{d-b-1}yp_6 | p_i \in K[x, z]\}$$

and the degrees of the $p_i's$ are fixed in such a way that the elements in V_d are homogeneous of degree d.

Proof. The generators of I_X form a Gröbner basis and so the initial ideal $in(I_X)$ of I_X with respect to the reverse lexicographic order is generated by w^2, y^2w, y^4, $x^{b-1}y^2$. Hence, the elements in V_d are in normal form with respect to I_X and so they are linearly independent. □

To describe the elements of $H^0(X, \mathcal{O}_X(d))$ for $2 \leq d \leq b$, we first define a suitable global section ξ of degree $-b+2$, and then we compute all the global sections by using ξ and the elements in $(R/I_X)_d$.

Definition 5.5. Let $\xi \in H^0(X, \mathcal{O}_X(-b+2))$ be the global section of X defined as

$$\xi = \frac{w}{x^{b-1}} = \frac{xz - y^2}{z^b}.$$

The global section ξ is well defined because for no closed point on C both x and z can be equal to 0, and because the two descriptions agree on the overlap (in fact $x^{b-1}(xz - y^2) - z^b w \in I_X$).

Proposition 5.6. *ξ verifies the following equalities*

1. $w\xi = (xz - y^2)\xi = 0$;
2. $x^{b-1}\xi = w$;
3. $z^b\xi = xz - y^2$.

Proof. They follow easily from the definition of ξ and from the knowledge of the ideal I_X. □

Proposition 5.7. *Let $2 \leq d \leq b$. Then, $H^0(X, \mathcal{O}_X(d)) = V_d$ where*

$$V_d = \{p_1 + yp_2 + (xz - y^2)p_3 + (xz - y^2)yp_4 + wp_5 + wyp_6 \\ + \xi(x^{d-1}z^d q_1 + x^{d-2}yz^{d-1}q_2) | p_i \in K[x, z], q_j \in K[x, z]\}$$

and the elements in V_d are homogeneous of degree d.

Proof. Of course, $V_d \subseteq H^0(X, \mathcal{O}_X(d))$.

As before, the elements $p_1 + yp_2 + (xz - y^2)p_3 + (xz - y^2)yp_4 + wp_5 + wyp_6$ are in normal form with respect to I_X and so they are linearly independent. Let $\pi : H^0_* \mathcal{O}_X \to H^1_* \mathcal{I}_X$. It is evident that $\xi \notin \ker(\pi) = R/I_X$ and so $\pi(\xi) \neq 0$. Hence $\pi(\xi(x^{d-1}z^d q_1 + x^{d-2}yz^{d-1}q_2)) = \pi(\xi)(x^{d-1}z^d q_1 + x^{d-2}yz^{d-1}q_2) \in R/\langle w, xz - y^2, x^{b-1}, z^b\rangle(-b+2)$. The generators of $\langle w, xz-y^2, x^{b-1}, z^b\rangle$ are a Gröbner basis and $x^{d-1}z^d q_1 + x^{d-2}yz^{d-1}q_2$ are in normal form with respect to the given Gröbner basis. Hence, they are linearly independent, and so V_d has dimension $\dim V_d = 4d+2b-2$, and the equality $H^0(X, \mathcal{O}_X(d)) = V_d$ holds. □

Now, we compute the degree 0 global sections of $H^0(X, \mathcal{N}_X)$ as

$$H^0(X, \mathcal{N}_X) = \ker(\mathrm{Hom}(F_1, H^0_*(X, \mathcal{O}_X)) \xrightarrow{\delta_1^\vee} \mathrm{Hom}(F_2, H^0(X, \mathcal{O}_X)))_0,$$

where, if $\varphi \in \mathrm{Hom}(F_1, H^0_*(X, \mathcal{O}_X))$ then $\delta_1^\vee(\varphi) = \varphi \circ \delta_1$.

Let $F_1 = \oplus_{i=1}^4 Re_i$ with $\deg(e_1) = 2, \deg(e_2) = 3, \deg(e_3) = 4, \deg(e_4) = b+1$, and assume $b \geq 4$. $\varphi \in \ker(\delta_1^\vee)$ if, and only if, the following system is satisfied:

$$\begin{cases} (xz - y^2)\varphi(e_1) - w\varphi(e_2) = 0 \\ (xz - y^2)\varphi(e_2) - w\varphi(e_3) = 0 \\ z^b\varphi(e_1) - x^{b-1}\varphi(e_2) + w\varphi(e_4) = 0 \\ z^b\varphi(e_2) - x^{b-1}\varphi(e_3) + (xz - y^2)\varphi(e_4) = 0. \end{cases} \tag{5.1}$$

with $\varphi(e_i) \in H^0(X, \mathcal{O}_X(\deg(e_i)))$.

To solve the system, we set

- $\varphi(e_1) = p_{11} + yp_{12} + (xz-y^2)p_{13} + wp_{15} + wyp_{16} + \xi(xz^2 q_{11} + yzq_{12})$ with $p_{1i} \in K[x, z], q_{1j} \in K[x, z]$ and $\deg(p_{11}) = 2, \deg(p_{12}) = \deg(p_{15}) = 1, \deg(p_{13}) = \deg(p_{16}) = 0, \deg(q_{11}) = b - 3$, $\deg(q_{12}) = b - 2$;
- $\varphi(e_2) = p_{21} + yp_{22} + (xz-y^2)p_{23} + (xz-y^2)yp_{24} + wp_{25} + wyp_{26} + \xi(x^2z^3q_{21} + xyz^2q_{22})$ with $p_{2i} \in K[x, z], q_{2j} \in K[x, z]$ and $\deg(p_{21}) = 3, \deg(p_{22}) = \deg(p_{25}) = 2, \deg(p_{23}) = \deg(p_{26}) = 1, \deg(p_{24}) = 0, \deg(q_{21}) = b - 4$, $\deg(q_{22}) = b - 3$;
- $\varphi(e_3) = p_{31} + yp_{32} + (xz-y^2)p_{33} + (xz-y^2)yp_{34} + wp_{35} + wyp_{36} + \xi(x^3z^4q_{31} + x^2yz^3q_{32})$ with $p_{3i} \in K[x, z], q_{3j} \in K[x, z]$ and $\deg(p_{31}) = 4, \deg(p_{32}) = \deg(p_{35}) = 3, \deg(p_{33}) = \deg(p_{36}) = 2, \deg(p_{34}) = 1, \deg(q_{31}) = b - 5$, $\deg(q_{32}) = b - 4$;

- $\varphi(e_4) = p_{41} + yp_{42} + (xz - y^2)p_{43} + (xz - y^2)yp_{44} + wxp_{45} + wyp_{46}$ with $p_{4i} \in K[x,z]$ and $\deg(p_{41}) = b+1, \deg(p_{42}) = b, \deg(p_{45}) = \deg(p_{43}) = \deg(p_{46}) = b-1, \deg(p_{44}) = b-2$.

Theorem 5.8. *With the notation as above, $h^0(X,\mathcal{N}_X) = 7 + 4b = 13 - 2g$, where g is the arithmetic genus of X.*

Proof. To get the claim, we have to compute the elements in $H^0(X,\mathcal{N}_X)$.
Claim: $\varphi \in H^0(X,\mathcal{N}_X)$ if, and only if, $\varphi(e_1) = 2wP_1 + 2wyP_2, \varphi(e_2) = (xz - y^2)P_1 + (xz-y^2)yP_2 + wP_3 + wyP_4, \varphi(e_3) = 2(xz-y^2)P_3 + 2(xz-y^2)yP_4, \varphi(e_4) = (x^{b-1}P_3 - z^bP_1) + y(x^{b-1}P_4 - z^bP_2) + (xz-y^2)P_5 + (xz-y^2)yP_6 + wxP_7 + wyP_8$ with $P_i \in K[x,z]$ for every i and $\deg(P_1) = \deg(P_4) = 1, \deg(P_2) = 0, \deg(P_3) = 2, \deg(P_5) = \deg(P_7) = \deg(P_8) = b-1, \deg(P_6) = b-2$.

If the claim holds, then we get the dimension of $H^0(X,\mathcal{N}_X)$ with an easy parameter count. Hence, we prove the Claim.

It is easy to check that if φ satisfies the given conditions, then $\varphi \in H^0(X,\mathcal{N}_X)$. Conversely, we solve the equations of the system (5.1) one at a time.

The first equation becomes

$$(xz - y^2)(p_{11} + yp_{12}) - w(p_{21} + yp_{22}) = 0.$$

As a R/I_X-module, the first syzygy module of $(xz - y^2, -w)$ is generated by

$$\begin{pmatrix} xz-y^2 & w & 0 & 0 & x^{b-1} \\ 0 & 0 & w & xz-y^2 & z^b \end{pmatrix}$$

and so we get the two equations

$$p_{11} + yp_{12} = 0 \quad p_{21} + yp_{22} = 0$$

because $p_{ij} \in K[x,z]$ and by degree argument.

Again as R/I_X-module, the first syzygy module of $(1,y)$ is generated by

$$\begin{pmatrix} y \\ -1 \end{pmatrix}$$

and so the solutions of two equations are $p_{11} = p_{12} = p_{21} = p_{22} = 0$.

The second equation, after substituting the computed solutions of the first one, becomes

$$w(p_{31} + yp_{32}) = 0.$$

We have that $p_{3i} \in K[x,z]$ and so $p_{31} + yp_{32} = 0$. Because of the same argument, and the knowledge of the first syzygy module of $(1,y)$, we get that the solutions of this equation are $p_{31} = p_{32} = 0$.

The third equation of the system (5.1) becomes

$$\begin{aligned}(xz - y^2)(x^{b-1}(p_{15} - p_{23}) + x^{b-1}y(p_{16} - p_{24}) + z^bp_{13} + xz^2q_{11} + yzq_{12}) \\ - w(x^{b-1}p_{25} + x^{b-1}yp_{26} - p_{41} - yp_{42} + x^2z^3q_{21} + xyz^2q_{22}) = 0.\end{aligned}$$

Because of the knowledge of the first syzygy module of $(xz - y^2, -w)$, from the previous equation we get the two ones

$$x^{b-1}(p_{15} - p_{23}) + x^{b-1}y(p_{16} - p_{24}) + z^b p_{13} + xz^2 q_{11} + yzq_{12} = x^{b-1}(r_1 + yr_2)$$

and

$$x^{b-1}p_{25} + x^{b-1}yp_{26} - p_{41} - yp_{42} + x^2z^3q_{21} + xyz^2q_{22} = z^b(r_1 + yr_2)$$

where $r_1 \in K[x,z]_1, r_2 \in K[x,z]_0$.

The first one can be rewritten as

$$x^{b-1}(p_{15} - p_{23} - r_1) + z^b p_{13} + xz^2 q_{11} + y(x^{b-1}(p_{16} - p_{24} - r_2) + zq_{12}) = 0.$$

From the knowledge of the first syzygy module of $(1, y)$ we get

$$x^{b-1}(p_{15} - p_{23} - r_1) + z^b p_{13} + xz^2 q_{11} = 0$$

and

$$x^{b-1}(p_{16} - p_{24} - r_2) + zq_{12} = 0.$$

Hence, $p_{15} = p_{23} + r_1, p_{13} = 0, q_{11} = 0, p_{16} = p_{24} + r_2, q_{12} = 0$.

The second equation can be rewritten as

$$x^{b-1}p_{25} - p_{41} + x^2z^3q_{21} - z^b r_1 + y(-p_{42} + x^{b-1}p_{26} + xz^2q_{22} - z^b r_2) = 0.$$

By using the same argument as before, we get

$$x^{b-1}p_{25} - p_{41} + x^2z^3q_{21} - z^b r_1 = 0$$

and

$$-p_{42} + x^{b-1}p_{26} + xz^2q_{22} - z^b r_2 = 0.$$

Hence, $p_{41} = x^{b-1}p_{25} + x^2z^3q_{21} - z^b r_1, p_{42} = x^{b-1}p_{26} + xz^2q_{22} - z^b r_2$.

The last equation of the system (5.1) becomes

$$\begin{aligned}(xz - y^2)(z^b(p_{23} - r_1) + 2x^2z^3q_{21} + y(z^b(p_{24} - r_2) + 2xz^2q_{22}))\\ - w(z^b(p_{33} - 2p_{25}) + x^{b-1}p_{35} + x^3z^4q_{31} + y(z^b(p_{34} - 2p_{26})\\ + x^{b-1}p_{36} + x^2z^3q_{32})) = 0.\end{aligned}$$

Then, there exist $r_3 \in K[x,z]_2, r_4 \in K[x,z]_1$ such that the two following equalities hold

$$z^b(p_{23} - r_1) + 2x^2z^3q_{21} + y(z^b(p_{24} - r_2) + 2xz^2q_{22}) = x^{b-1}(r_3 + yr_4)$$

and

$$\begin{aligned}z^b(p_{33} - 2p_{25}) + x^{b-1}p_{35} + x^3z^4q_{31} + y(z^b(p_{34} - 2p_{26}) + x^{b-1}p_{36} + x^2z^3q_{32})\\ = z^b(r_3 + yr_4).\end{aligned}$$

From the first equation, we get

$$z^b(p_{23} - r_1) - x^{b-1}r_3 + 2x^2z^3q_{21} = 0$$

and

$$z^b(p_{24} - r_2) - x^{b-1}r_4 + 2xz^2q_{22} = 0.$$

Hence, $r_1 = p_{23}, r_2 = p_{24}, r_3 = r_4 = 0, q_{21} = q_{22} = 0$. From the second equation, we get

$$z^b(p_{33} - 2p_{25}) + x^{b-1}p_{35} + x^3z^4q_{31} = 0$$

and

$$z^b(p_{34} - 2p_{26}) + x^{b-1}p_{36} + x^2z^3q_{32} = 0.$$

As before, we deduce that $p_{33} = 2p_{25}, p_{34} = 2p_{26}, p_{35} = p_{36} = 0, q_{31} = q_{32} = 0$.

Summarizing, we obtain the following:

1. $\varphi(e_1) = 2w(p_{23} + yp_{24})$,
2. $\varphi(e_2) = (xz - y^2)(p_{23} + yp_{24}) + w(p_{25} + yp_{26})$;
3. $\varphi(e_3) = 2(xz - y^2)(p_{25} + yp_{26})$,
4. $\varphi(e_4) = x^{b-1}(p_{25}+yp_{26})-z^b(p_{23}+yp_{24})+(xz-y^2)(p_{43}+yp_{44})+w(xp_{45}+yp_{46})$,

and the claim follows with the obvious substitutions. □

Remark 5.9. We computed $h^0(X, \mathcal{N}_X)$ by using the function `<normal_sheaf` of the computer algebra software Macaulay (see [2]), in the cases $0 \leq b \leq 3$.

If $b = 3$, or equivalently $g = -3$, we get that the dimension of the degree 0 global sections of the normal sheaf of X with saturated ideal

$$I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, x^2(xz - y^2) - z^3w\rangle$$

is equal to $h^0(X, \mathcal{N}_X) = 19 = 13 - 2g$.

If $b = 2$, i.e., $g = -1$, the dimension of the degree 0 global sections of the normal sheaf of the double conic X defined by the ideal

$$I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, x(xz - y^2) - z^2w\rangle$$

is equal to $h^0(X, \mathcal{N}_X) = 16 \neq 13 - 2g$.

If $b = 1$, i.e., $g = 1$, the double conic X is defined by the ideal

$$I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, (xz - y^2) - zw\rangle$$

and $h^0(X, \mathcal{N}_X) = 16 \neq 13 - 2g$.

At last, if $b = 0$, i.e., $g = 3$, then the double conic X is defined by the ideal

$$I_X = \langle w, (xz - y^2)^2\rangle$$

and $h^0(X, \mathcal{N}_X) = 17 \neq 13 - 2g$.

5.2. Case g even, i.e., $a = 2b + 1$

In this subsection, we repeat what we did in the previous subsection, by sketching the main differences.

The running notation of the subsection are the following. As before, we set $R = K[x, y, z, w]$ and $C \subset \mathbb{P}^3 = \mathrm{Proj}(R)$ is the conic defined by the ideal $I_C = (xz-y^2, w)$. C is isomorphic to $\mathbb{P}^1$ via $j : \mathbb{P}^1 \to C$ defined as $j(t : u) = (t^2 : tu : u^2 : 0)$. We set $\mu : \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-3 + 2b)$ defined as $\mu = (u^{2b+1}, t^{2b-1})$.

Proposition 5.10. *If X is the doubling of C associated to μ, then*

1. $I_X = \langle w^2, w(xz-y^2), (xz-y^2)^2, x^b(xz-y^2) - yz^b w, x^{b-1}y(xz-y^2) - z^{b+1}w\rangle$;
2. *the generators of I_X are a Gröbner basis with respect to the reverse lexicographic order;*
3. *the minimal free resolution of I_X is*

$$0 \to R^2(-b-4) \xrightarrow{\delta_2} \begin{matrix} R(-4) \\ \oplus \\ R(-5) \\ \oplus \\ R^4(-b-3) \end{matrix} \xrightarrow{\delta_1} \begin{matrix} R(-2) \\ \oplus \\ R(-3) \\ \oplus \\ R(-4) \\ \oplus \\ R^2(-b-2) \end{matrix} \longrightarrow I_X \to 0,$$

where the maps δ_1 and δ_2 will be described in the proof.

Proof. The first syzygy module of $H^0_*(j_*\mu) = (yz^b, x^b)$ is generated by

$$N = \begin{pmatrix} x^b & x^{b-1}y \\ -yz^b & -z^{b+1} \end{pmatrix}.$$

Let M be the matrix we get by reading N over R and not over R/I_C. Then, the saturated ideal of X is

$$I_X = \langle w^2, w(xz-y^2), (xz-y^2)^2, x^b(xz-y^2) - yz^b w, x^{b-1}y(xz-y^2) - z^{b+1}w\rangle.$$

The check on S-polynomials holds on the generators of I_X and so they are a Gröbner basis.

To compute the first syzygy module of I_X we proceed as in the proof of Proposition 5.1. The computation is quite similar and uses the same ideas. Hence, we write only the maps δ_1 and δ_2:

$$\delta_1 = \begin{pmatrix} xz-y^2 & 0 & -yz^b & 0 & 0 & z^{b+1} \\ -w & xz-y^2 & x^b & z^b & 0 & -x^{b-1}y \\ 0 & -w & 0 & 0 & -x^{b-1} & 0 \\ 0 & 0 & -w & -y & z & 0 \\ 0 & 0 & 0 & x & -y & w \end{pmatrix}$$

and

$$\delta_2 = \begin{pmatrix} z^b & 0 \\ 0 & x^{b-1} \\ -y & -z \\ w & 0 \\ 0 & -w \\ -x & -y \end{pmatrix}.$$

□

Proposition 5.11. *X has genus $g(X) = 2 - 2b$, the Hartshorne-Rao function of X is equal to*

$$h^1\mathcal{I}_X(j) = \begin{cases} 2(b+j-1) & \text{if } -b+1 \leq j \leq 0 \\ 2b-1 & \text{if } j = 1 \\ 2(b-j+1) & \text{if } 2 \leq j \leq b+1 \\ 0 & \text{otherwise} \end{cases}$$

and $\delta_2^\vee$ is a presentation matrix for $H^1_*\mathcal{I}_X$.

Proof. The proof of the first two statements rests on results from Section 3. The last statement follows from the minimality of X in its biliaison class. □

Proposition 5.12. *$h^0(X, \mathcal{O}_X(d)) = 4d + 2b - 1$ for $d \geq 2$.*

Proof. See Proposition 5.3. □

As before, we describe the elements of $H^0(X, \mathcal{O}_X(d))$ for every $d \geq 2$. At first, we describe the elements of $H^0(X, \mathcal{O}_X(d))$ for $d \geq b+1$, because, in the considered range, we have $(R/I_X)_d = H^0(X, \mathcal{O}_X(d))$, and so it holds

Proposition 5.13. *Let $d \geq b+1$. Then, $H^0(X, \mathcal{O}_X(d))$ is equal to*

$$\{p_1 + yp_2 + (xz - y^2)p_3 + (xz - y^2)yp_4 + wx^{d-b-1}p_5 + wx^{d-b-1}yp_6 | p_i \in K[x, z]\}$$

where the degrees of the p_i' s are fixed in such a way that the elements are homogeneous of degree d.

To describe the elements of $H^0(X, \mathcal{O}_X(d))$ for $2 \leq d \leq b$, this time we need two suitable global sections ξ_1, ξ_2 of degree $-b+2$, and then we compute all the global sections by using ξ_1, ξ_2 and the elements in $(R/I_X)_d$.

Definition 5.14. Let $\xi_1, \xi_2 \in H^0(X, \mathcal{O}_X(-b+2))$ be the global section of X defined as

$$\xi_1 = \frac{xz - y^2}{z^b} = \frac{yw}{x^b} \text{ and } \xi_2 = \frac{y(xz - y^2)}{z^{b+1}} = \frac{w}{x^{b-1}}.$$

The global sections ξ_1 and ξ_2 are well defined because for no closed point on C both x and z can be equal to 0, and because the two definitions agree on the overlap (see the last two generators of I_X.)

Proposition 5.15. *ξ_1 and ξ_2 verify the following equalities*

1. $w\xi_1 = (xz - y^2)\xi_1 = w\xi_2 = (xz - y^2)\xi_2 = 0$;
2. $x^b\xi_1 = yw, z^b\xi_1 = xz - y^2$;
3. $x^{b-1}\xi_2 = w, yz^b\xi_2 = x(xz - y^2), z^{b+1}\xi_2 = y(xz - y^2)$;
4. $x\xi_1 = y\xi_2, y\xi_1 = z\xi_2$.

Proof. The equalities easily follow from the definition of ξ_1, ξ_2 and from the knowledge of the ideal I_X. □

Proposition 5.16. *Let $2 \leq d \leq b$. Then $H^0(X, \mathcal{O}_X(d)) = V_d$ where*

$$V_d = \{p_1 + yp_2 + (xz - y^2)p_3 + (xz - y^2)yp_4 + wp_5 + wyp_6 \\ + x^{d-2}z^{d-1}\xi_2(zq_1 + yq_2) \mid p_i \in K[x, z], q_j \in K[x, z]\}$$

and the elements in V_d are homogeneous of degree d.

Proof. We can apply the same argument as in Proposition 5.7, with the only difference that

$$\{f \in R | f\xi_2 \in R/I_X\} = \langle w, xz - y^2, x^{b-1}, yz^b, z^{b+1}\rangle$$

as can be easily computed via computer algebra techniques. □

Remark 5.17. Thanks to the relations of previous Proposition 5.15, we can write the elements in $H^0(X, \mathcal{O}_X(d))$ without using ξ_1. Analogously, one can write them using ξ_1 but not ξ_2. The choice of using only one between ξ_1, ξ_2 allows to simplify the following computations.

We want to compute the degree 0 global sections of the normal sheaf of X as

$$H^0(X, \mathcal{N}_X) = \ker(\mathrm{Hom}(F_1, H^0_*(X, \mathcal{O}_X)) \xrightarrow{\delta_1^\vee} \mathrm{Hom}(F_2, H^0_*(X, \mathcal{O}_X)))_0$$

where $F_1 = R(-2) \oplus R(-3) \oplus R(-4) \oplus R^2(-b-2)$ and $F_2 = R(-4) \oplus R(-5) \oplus R^4(-b-3)$ and $\delta_1^\vee$ is the dual of $\delta_1 : F_2 \to F_1$.

To this end, let $\varphi \in \mathrm{Hom}(F_1, H^0_*(X, \mathcal{O}_X))$ be a degree 0 map. Then, if $F_1 = \oplus_{i=1}^5 Re_i$ with $\deg(e_1) = 2, \deg(e_2) = 3, \deg(e_3) = 4, \deg(e_4) = \deg(e_5) = b + 2$, we have that $\varphi(e_i) \in H^0(X, \mathcal{O}_X(\deg(e_i)))$, and so, if we assume that $b \geq 4$, we can set

- $\varphi(e_1) = p_{11} + yp_{12} + (xz - y^2)p_{13} + wp_{15} + wyp_{16} + \xi_2(z^2q_{11} + yzq_{12})$, with $\deg(p_{11}) = 2, \deg(p_{12}) = \deg(p_{15}) = 1, \deg(p_{13}) = \deg(p_{16}) = 0, \deg(q_{11}) = \deg(q_{12}) = b - 2$;
- $\varphi(e_2) = p_{21} + yp_{22} + (xz - y^2)p_{23} + (xz - y^2)yp_{24} + wp_{25} + wyp_{26} + \xi_2(xz^3q_{21} + xyz^2q_{22})$, with $\deg(p_{21}) = 3, \deg(p_{22}) = \deg(p_{25}) = 2, \deg(p_{23}) = \deg(p_{26}) = 1, \deg(p_{24}) = 0, \deg(q_{21}) = \deg(q_{22}) = b - 3$;
- $\varphi(e_3) = p_{31} + yp_{32} + (xz - y^2)p_{33} + (xz - y^2)yp_{34} + wp_{35} + wyp_{36} + \xi_2(x^2z^4q_{31} + x^2yz^3q_{32})$, with $\deg(p_{31}) = 4, \deg(p_{32}) = \deg(p_{35}) = 3, \deg(p_{33}) = \deg(p_{36}) = 2, \deg(p_{34}) = 1, \deg(q_{31}) = \deg(q_{32}) = b - 4$;
- $\varphi(e_4) = p_{41} + yp_{42} + (xz - y^2)p_{43} + (xz - y^2)yp_{44} + wxp_{45} + wxyp_{46}$, with $\deg(p_{41}) = b + 2, \deg(p_{42}) = b + 1, \deg(p_{43}) = \deg(p_{45}) = b, \deg(p_{44}) = \deg(p_{46}) = b - 1$;
- $\varphi(e_5) = p_{51} + yp_{52} + (xz - y^2)p_{53} + (xz - y^2)yp_{54} + wxp_{55} + wxyp_{56}$, with $\deg(p_{51}) = b + 2, \deg(p_{52}) = b + 1, \deg(p_{53}) = \deg(p_{55}) = b, \deg(p_{54}) = \deg(p_{56}) = b - 1$.

Of course, $\varphi \in H^0(X, \mathcal{N}_X)$ if, and only if, $\delta_1^\vee(\varphi) = \varphi \circ \delta_1 = 0$, and so we get the following system

$$\begin{cases} (xz - y^2)\varphi(e_1) - w\varphi(e_2) = 0 \\ (xz - y^2)\varphi(e_2) - w\varphi(e_3) = 0 \\ -yz^b\varphi(e_1) + x^b\varphi(e_2) - w\varphi(e_4) = 0 \\ z^b\varphi(e_2) - y\varphi(e_4) + x\varphi(e_5) = 0 \\ -x^{b-1}\varphi(e_3) + z\varphi(e_4) - y\varphi(e_5) = 0 \\ z^{b+1}\varphi(e_1) - x^{b-1}y\varphi(e_2) + w\varphi(e_5) = 0. \end{cases} \tag{5.2}$$

A technical result in the computation of $H^0(X, \mathcal{N}_X)$ is the knowledge of the generators of two syzygy modules. In particular, it holds

Lemma 5.18. *In R/I_X, the first syzygy module of $(xz - y^2, -w)$ is generated by*

$$\begin{pmatrix} xz - y^2 & w & 0 & 0 & x^b & x^{b-1}y \\ 0 & 0 & w & xz - y^2 & yz^b & z^{b+1} \end{pmatrix}$$

while the first syzygy module of $(1, y)$ is minimally generated by

$$\begin{pmatrix} y \\ -1 \end{pmatrix}.$$

The proof is based on standard Gröbner bases arguments.

Thanks to the previous Lemma, we can solve system (5.2), one equation at a time, and we get

Theorem 5.19. *With the notation as above, $h^0(X, \mathcal{N}_X) = 9 + 4b = 13 - 2g$.*

Proof. To prove the statement, we have to compute the elements in $H^0(X, \mathcal{N}_X)$. *Claim:* $\varphi \in H^0(X, \mathcal{N}_X)$ if, and only if, there exist $\alpha, \beta \in K$, and $P_1, \ldots, P_8 \in K[x, z]$, of suitable degrees, such that

- $\varphi(e_1) = 2w(P_1 + yP_2)$;
- $\varphi(e_2) = (xz - y^2)(P_1 + yP_2) + w(\alpha x^2 + zP_3 + yP_4)$;
- $\varphi(e_3) = 2(xz - y^2)(\alpha x^2 + zP_3 + yP_4)$;
- $\varphi(e_4) = x^b(\alpha x^2 + zP_3 + yP_4) - yz^b(P_1 + yP_2) + (xz - y^2)(xP_5 + yP_6 + x^{b-1}P_3) + xw(xP_7 + yP_8 + \beta z^b)$;
- $\varphi(e_5) = x^{b-1}y(\alpha x^2 + zP_3 + yP_4) - z^{b+1}(P_1 + yP_2) + (xz - y^2)(yP_5 + zP_6 - x^{b-1}P_4) + xw(yP_7 + zP_8 - \alpha z^b)$.

If the claim holds, then we can compute $h^0(X, \mathcal{N}_X)$ with an easy parameter count. Hence, we prove the claim. Its proof is a quite long computation where we use the same ideas as in the proof of Theorem 5.8. □

Remark 5.20. Now, we consider the cases not covered by Theorem 5.19, namely $1 \leq b \leq 3$. We consider the double conic defined by the ideal

$$I_X = \langle w^2, w(xz - y^2), (xz - y^2)^2, x^b(xz - y^2) - yz^b w, x^{b-1}y(xz - y^2) - z^{b+1}w \rangle$$

with $b = 1, 2, 3$, and we compute $h^0(X, \mathcal{N}_X)$ by using Macaulay (see [2]).

If $b = 3$, we get $h^0(X, \mathcal{N}_X) = 21 = 13 - 2g$, because $g = 2 - 2b = -4$.
If $b = 2$, we get $h^0(X, \mathcal{N}_X) = 17 = 13 - 2g$, because $g = -2$.
If $b = 1$, then we get $H^0(X, \mathcal{N}_X) = 16 \neq 13 - 2g$, because, in this case, $g = 0$.

5.3. Case $X \subseteq \mathbb{P}^n, n \geq 4$

Now, we suppose that X is a suitable double conic in $\mathbb{P}^n, n \geq 4$.

Theorem 5.21. *Let $C \subset \mathbb{P}^n$ be the conic defined by the ideal*

$$I_C = \langle x_0 x_2 - x_1^2, x_3, \ldots, x_n \rangle,$$

and let $j : \mathbb{P}^1 \to C$ be the isomorphism defined as $j(t : u) = (t^2 : tu : u^2 : 0 : \cdots : 0)$. Let $\mu : \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-4 + a), a \geq 5$, be the map defined as $\mu = (u^a, t^{a-2}, 0, \ldots, 0)$, and let X be the double structure on C associated to μ and j. Then, $h^0(X, \mathcal{N}_X) \leq (n-1)(5-g) + 3$.

Proof. $\mathcal{O}_{\mathbb{P}^1}^{n-2}(-2)$ is contained in the kernel of μ. Hence, $x_3, \ldots, x_n \in I_X$, i.e., X is degenerate and it is contained in the linear space L of dimension 3. Then, we can consider both the normal sheaf $\mathcal{N}_{X,L}$ of X in L, and the normal sheaf $\mathcal{N}_X$ of X in $\mathbb{P}^n$. They are related via the exact sequence

$$0 \to \mathcal{N}_{X,L} \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{O}_X(1)^{n-3}.$$

In particular, we have the inequality

$$h^0(X, \mathcal{N}_X) \leq h^0(X, \mathcal{N}_{X,L}) + (n-3) h^0(X, \mathcal{O}_X(1)).$$

By hypothesis, $a \geq 5$ and so the arithmetic genus g of X satisfies $g \leq -2$. By Theorems 5.8, 5.19 and the remarks after them, $h^0(X, \mathcal{N}_{X,L}) = 13 - 2g$, while $h^0(X, \mathcal{O}_X(1)) = \dim_K (R/I_X)_1 + h^1 \mathcal{I}_X(1) = 4 + a - 2 = 5 - g$ as proved in Propositions 3.9. Then,

$$h^0(X, \mathcal{N}_X) \leq 13 - 2g + (n-3)(5-g) = (n-1)(5-g) + 3. \qquad \square$$

5.4. Remarks on the Hilbert schemes $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^n)$

In this last subsection, we use the previous results to get information on the irreducible components containing the double structures of genus g on conics.

Theorem 5.22. *If $g \leq -2$, then $\overline{H(4, g, n)}$ is a generically smooth irreducible component of $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^n)$ of dimension $(n-1)(5-g) + 3$.*

Proof. By Corollary 2.13, every irreducible component containing $\overline{H(4, g, n)}$ has dimension greater than or equal to $(n-1)(5-g)+3$, but the tangent space to the Hilbert scheme $\mathcal{H}ilb_{4t+1-g}(\mathbb{P}^n)$ at the double conic described in Theorem 5.21 has dimension lesser than or equal to $(n-1)(5-g)+3$. Hence, the point corresponding to the double conic considered in Theorem 5.21 is smooth, $\overline{H(4, g, n)}$ is irreducible of dimension $\dim \overline{H(4, g, n)} = (n-1)(5-g)+3$ and the point corresponding to the double conic considered in Theorem 5.21 is smooth, i.e., $\overline{H(4, g, n)}$ is generically smooth. $\square$

Now, we add some remarks to $\overline{H(4, g, 3)}$ for $g \geq -1$.

Proposition 5.23. *If $g = -1$, the general element of $\overline{H(4,-1,3)}$ is the union of two smooth conics without common points, and a double structure on a smooth conic is a smooth point of $\overline{H(4,-1,3)}$.*

Proof. By a simple parameter count, the family of two disjoint conics has dimension 16, which is equal to the dimension of the tangent space to $\overline{H(4,-1,3)}$ at the double conic considered in Theorem 5.8 and in the subsequent Remark. The claim follows if we exhibit a family whose general element is a disjoint union of two conics, and whose special fiber is the considered double structure. The ideal

$$\langle w, xz - y^2\rangle \cap \langle w + tx, tz^2 + xz - y^2\rangle \subseteq K[x, y, z, w, t]$$

gives a flat family over $\mathbb{A}^1$ with the required properties. □

Proposition 5.24. *If $g = 0$, the general element of $\overline{H(4,0,3)}$ is a rational quartic curve, and a double structure on a smooth conic is a smooth point of $\overline{H(4,0,3)}$.*

Proof. By a simple parameter count, the family of the rational quartic curves has dimension 16, which is equal to the dimension of the tangent space to $\overline{H(4,0,3)}$ at the double conic considered in Theorem 5.19 and in the subsequent Remark. The claim follows if we exhibit a family whose general element is a rational quartic curve, and whose special fiber is the considered double structure. The ideal

$$\langle w^2 + t(xy - zw), y(xz - y^2) - z^2w\rangle : \langle xy - zw, y^2, yw, w^2\rangle \subseteq K[x, y, z, w, t]$$

gives a flat family over $\mathbb{A}^1$ with the required properties. In fact, a general quartic curve is linked to two skew lines or to a double line of genus -1 via a complete intersection of type $(2, 3)$. The general element of the family is the residual intersection of a double line of genus -1 on a smooth quadric surface, while the special element is the residual intersection to the same double line on a double plane. □

Proposition 5.25. *If $g = 1$, the general element of $\overline{H(4,1,3)}$ is the complete intersection of two quadric surfaces, and a double structure on a smooth conic is a smooth point of $\overline{H(4,1,3)}$.*

Proof. By a simple parameter count, the family of the complete intersections of two quadric surfaces has dimension 16, which is equal to the dimension of the tangent space to $\overline{H(4,1,3)}$ at the double conic considered in Theorem 5.8 and in the subsequent remark. The claim follows because it is easy to check that the ideal of the considered double conic is the complete intersection of w^2 and $xz-y^2-zw$. □

Proposition 5.26. *If $g = 3$, the general element of $\overline{H(4,3,3)}$ is a plane quartic curve, and a double structure on a smooth conic is a smooth point of $\overline{H(4,3,3)}$.*

Proof. By a simple parameter count, the family of the plane quartic curves has dimension 17, which is equal to the dimension of the tangent space to $\overline{H(4,3,3)}$ at the double conic considered in Theorem 5.8 and in the subsequent remark. The claim follows because the double conic we considered is a plane quartic curve. □

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Part III

The Schottky Problem

This part has been coordinated by Raquel Mallavibarrena with the advice of José María Muñoz-Porras. It is devoted to the Schottky problem and it contains a survey and two research articles.

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Survey on the Schottky Problem

Esteban Gómez González and José M. Muñoz Porras

Dedicated to the memory of Prof. Federico Gaeta

Abstract. There are excellent surveys on the history of the Schottky problem like [16]. Our aim in this paper is to expose the approaches to the Schottky problem more directly related to the recent results proved by I. Krichever and S. Grushevsky ([8, 9, 12, 13]). That is, we will expose the approaches related with the existence of trisecants, the KP equation and the Γ_{00}-conjecture.

Mathematics Subject Classification (2000). Primary 14H42; Secondary 14K25, 14H40.

Keywords. Schootky problem, Jacobians, theta functions.

1. The statement of the Schottky problem

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$ (p.p.a.v.) of dimension g, and let $\mathcal{M}_g$ be the moduli space of smooth algebraically irreducible curves of genus g over $\mathbb{C}$. There exists a morphism of schemes:

$$J_g \colon \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

called the Jacobi map which is defined by:

$$J_g(C) = (J(C), W_{g-1})$$

where $J(C)$ is the Jacobian of the curve C and W_{g-1} is its theta divisor. The injectivity of J_g is the Torelli Theorem ([1, 14, 17]). One can define the Jacobian locus, $\mathcal{J}_g$, in dimension g as the image of the Jacobi map, and the closed Jacobian locus, $\overline{\mathcal{J}}_g$, as the closure of $\mathcal{J}_g$ in $\mathcal{A}_g$. It is well known that $\overline{\mathcal{J}}_g$ represents the set of p.p.a.v. which are of the form

$$(J(C_1) \times \cdots \times J(C_r), \Theta = \pi_1^* W_{g_1-1} + \cdots + \pi_r^* W_{g_r-1})$$

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where $(J(C_i), W_{g_i-1})$ is the polarized Jacobian of the curve C_i of genus g_i, $g = g_1 + \cdots + g_r$ and $\pi_i \colon J(C_1) \times \cdots \times J(C_r) \to J(C_i)$ is the natural projection.

One has that

$$\dim \overline{\mathcal{J}}_g = 3g - 3 \quad (\text{for } g > 1)$$
$$\dim \mathcal{A}_g = g(g+1)/2 \ .$$

Then for $g \geq 4$, $\overline{\mathcal{J}}_g$ is a proper closed subscheme of $\mathcal{A}_g$ ($\overline{\mathcal{J}}_g \subsetneq \mathcal{A}_g$).

From a general point of view we could state the Schottky problem as the problem of characterize $\overline{\mathcal{J}}_g$ as a subscheme of $\mathcal{A}_g$. There are several approaches to this problem characterizing $\overline{\mathcal{J}}_g$ in terms of geometric properties. A more restrictive statement should be to find explicit algebraic or differential equations for θ-functions which characterize $\overline{\mathcal{J}}_g$ as a subscheme of $\mathcal{A}_g$.

Characterizing $\overline{\mathcal{J}}_g$ in terms of algebraic equations goes back to the original paper of Schottky-Jung ([18]) and there have been proved partial results by R. Donagi ([3, 4]) and B. van Geemen ([6]).

The problem of characterizing $\overline{\mathcal{J}}_g$ in terms of differential equation was solved by T. Shiota ([19]) and M. Mulase ([15]). This approach is related to the geometric characterization of Jacobian in terms of the existence of trisecant lines in its Kummer variety (R. Gunning [10], G. Welters [20], I. Krichever [12, 13]).

2. Characterization of Jacobians in terms of the existence of trisecants

Let (X, Θ) be an indecomposable p.p.a.v of dimension g (we assume that Θ is symmetric and irreducible). One has a natural morphism of schemes given by second order theta functions:

$$\phi_X \colon X \longrightarrow \mathbb{P}^N = \operatorname{Proj} S^\bullet \left(H^0(X, \mathcal{O}_X(2\Theta))^* \right) \ , \quad N = 2^g - 1 \ .$$

The image of ϕ_X is the Kummer variety $K(X) = X/\{\pm 1\}$ of X, whose singular points are the images under ϕ_X of the 2-torsion points of X.

Let $Y \subset X$ be an artinian subscheme of length 3 and define:

$$V_Y = \left\{ 2\xi \ : \ \xi + Y \subset \phi_X^{-1}(l) \ , \text{ for some line } l \subset \mathbb{P}^N \right\} .$$

Welters ([20]) proved that if V_Y has positive dimension at some point and for some subscheme Y, then V_Y is a smooth irreducible algebraic curve and (X, Θ) is its polarized Jacobian. This result generalizes and clarifies the previous characterization proved by R. Gunning ([10]) in terms of the existence of trisecant lines for the Kummer variety.

This result justifies the following question: Can be characterized the Jacobian by the existence of one trisecant line or one inflectionary tangent at non singular point to the Kummer variety? This conjecture known as the "trisecant conjecture" has been recently proved by I. Krichever ([12, 13]).

The Novikov conjecture, proved by T. Shiota ([19]) can be stated as follows: (X, Θ) is a Jacobian if and only if there exists a subscheme $Y = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^3 \hookrightarrow X$

for which length$(\mathcal{O}_{Y,0}) \geq 4$. This statement is equivalent to say that the theta function $\theta(z,\tau)$ associated with (X,Θ) satisfies the KP equation in Hirota's form (see [2] for the equivalence between both statements): There exist constant vector fields D_1, D_2, D_3 on $\mathbb{C}^g$ and a constat $d \in \mathbb{C}$ such that

$$\begin{aligned} D_1^4\theta\cdot\theta - 4D_1^3\theta\cdot D_1\theta + 3D_1^2\theta\cdot D_1^2\theta + 3D_2^2\theta\cdot\theta \\ - 3D_2\theta\cdot D_2\theta - 3D_1D_3\theta\cdot\theta + 3D_3\theta\cdot D_1\theta - d\theta\cdot\theta \ = 0\ . \end{aligned} \tag{2.1}$$

The KP equation can be re-stated in terms of second order theta functions in the following way: A basis of $H^0(X,\mathcal{O}_X(2\Theta))$ is given by the set of second order theta functions:

$$\theta_2[\sigma](z) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i\Big((n+\sigma/2)^t\tau(n+\sigma/2) + 2(n+\sigma/2)^t z\Big)$$

with $z \in \mathbb{C}^g$, τ is the matrix period of (X,Θ), and $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$.

For a point $x \in X$, let us denote by $\vec{\theta}_2(x)$ the vector of $H^0(X,\mathcal{O}_X(2\Theta))$ with coordinates $\{\theta_2[\sigma](x)\}$. Then, the KP equation (2.1) for the theta function is equivalent to say that there exist constant vector fields D_1, D_2, D_3 on $\mathbb{C}^g$ and a constant $d \in \mathbb{C}$ such that:

$$\Big(D_1^4 - D_1D_3 + \frac{3}{4}D_2^2 + d\Big)\vec{\theta}_2(0) \ = 0 \tag{2.2}$$

3. The Γ_{00}-conjecture

The third approach to the Schottky problem which we will describe is related with a conjecture stated by van Geemen and van der Geer ([7]). Let us define the subspace Γ_{00} of $H^0(X,\mathcal{O}_X(2\Theta))$ by

$$\Gamma_{00} = \big\{s \in H^0(X,\mathcal{O}_X(2\Theta)) \ : \ m_o(s) \geq 4\big\}$$

where $m_0(s)$ denotes the multiplicity of vanishing of a section s at the origin $0 \in X$. Then we can define the following subscheme of X:

$$F_X = \{x \in X \ : \ s(x) = 0 \ \text{ for all } s \in \Gamma_{00}\}$$

Now, we can state

Conjecture 3.1. (van Geemen, van der Geer) *For $g \geq 2$, (X,Θ) is a Jacobian if and only if* $\dim F_X \geq 2$.

This conjecture has been proved by E. Izadi ([11]) in the case $g = 4$. When $X = J(C)$ is a Jacobian of a smooth curve, it is well known that F_X is the surface $C - C \subset J(C)$ for $g \geq 5$ ([7]). F. Gaeta in [5] studied the surface $C - C$ from the point of view of the classical theory of correspondences and in his paper there are some interesting remarks about the connection between the geometry of $C - C$ and the Schottky-Jung equations.

The points of F_X can be described as another way. Let S_1 be the subspace of $H^0(X, \mathcal{O}_X(2\Theta))$ spanned by the vectors $\vec{\theta}_2(0)$ and $\partial_i\partial_j\vec{\theta}_2(0)$ with $i \leq j$, where $\partial_i\partial_j\vec{\theta}_2(0)$ designates the vector with coordinates

$$\left\{\frac{\partial^2\theta_2[\sigma](z)}{\partial z_i \partial z_j}(0)\right\}_{\sigma\in(\mathbb{Z}/2\mathbb{Z})^g}.$$

The subspace S_1 is the orthogonal to Γ_{00} with respect to the natural pairing of second order theta functions. Then the points of F_X can be described as follows (see [7]): $x \in F_X$ if and only if there exist $\lambda, \mu_{ij} \in \mathbb{C}$ $(i \leq j)$ such that

$$\vec{\theta}_2(x) = \lambda \cdot \vec{\theta}_2(0) + \sum_{i\leq j} \mu_{ij} \cdot \partial_i\partial_j \vec{\theta}_2(0). \tag{3.1}$$

S. Grushevsky has proved a particular case of the Γ_{00}-conjecture (see the paper of S. Grushevsky in this volume): (X, Θ) is a Jacobian if and only if the equation (3.1) is satisfied for a symmetric matrix $\mu = (\mu_{ij})$ of rank at most two.

Let us describe a geometric interpretation of this conjecture and its connection with the approach to the Schottky problem in terms of the KP equation. Let $K(X) \subset \mathbb{P}^N$ be the Kummer variety of X and $p_0 = \phi_X(0)$ the image of the origin of X. Obviously p_0 is a singular point of $K(X)$ and its Zariski tangent space can be canonically identified with

$$T_{p_0}(K(X)) \simeq S^2T_0(X) \simeq S^2H^0(\Theta, \mathcal{O}_\Theta(\Theta)) \subset H^0(\Theta, \mathcal{O}_\Theta(2\Theta))$$

where $S^2(\ )$ means the second symmetric product. If we consider the natural restriction map:

$$\alpha\colon H^0(X, \mathcal{O}_X(2\Theta)) \longrightarrow H^0(\Theta, \mathcal{O}_\Theta(2\Theta))$$

one can easily show that

$$\alpha^{-1}\Big(T_{p_0}(K(X))\Big) = S_1$$

Then the subspace $\mathbb{P}(S_1) \subset \mathbb{P}^N$ is precisely the projective closure in $\mathbb{P}^N$ of the Zariski tangent space to $K(X)$ at p_0 and one has:

$$F_X = \phi_X^{-1}\Big(K(X) \cap \mathbb{P}(S_1)\Big)$$

If we denote $\mathbb{P}^{g-1} = \mathbb{P}(H^0(\Theta, \mathcal{O}_\Theta(\Theta)))$, then the image of the natural map

$$\mathbb{P}^{g-1} \times \mathbb{P}^{g-1} \longrightarrow \mathbb{P}(S^2H^0(\Theta, \mathcal{O}_\Theta(\Theta))) \subset \mathbb{P}(H^0(\Theta, \mathcal{O}_\Theta(2\Theta)) = \mathbb{P}^{N-1}$$

is a subscheme X_{2g-2} of $\mathbb{P}^{N-1}$ and the inverse image, $X_{2g-1} = \bar{\alpha}^{-1}(X_{2g-2})$, with respect the rational map $\bar{\alpha}\colon \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ induced by α, is a subscheme of dimension $2g-1$.

The result of S. Grushevsky is equivalent to say that (X, Θ) is a Jacobian if and only if

$$K(X) \cap X_{2g-1} \neq \{p_0\}$$

The proof of the result of S. Grushevsky is based on the previous papers of I. Krichever ([12, 13]) on the trisecant conjecture.

Using the identity (3.1) one can compute the points of F_X centered at the origin with values in $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^n)$. The existence of a point of F_X centered at origin with values in $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^5)$ is equivalent to the existence of a constant vector fields D_1, D_2, D_3, a constant $d \in \mathbb{C}$ and a symmetric matrix (μ_{ij}) such that the following identity is verified:

$$\Big(\frac{1}{24}D_1^4 + \frac{1}{2}D_2^2 - D_1D_3\Big)\vec{\theta}_2(0) = d \cdot \vec{\theta}_2(0) + \sum_{i\leq j} \mu_{ij} \cdot \partial_i\partial_j \vec{\theta}_2(0) \tag{3.2}$$

A necessary and sufficient condition for the identity (3.2) to be equivalent to the KP equation in the form (2.2) is that

$$\sum_{i\leq j} \mu_{ij} \cdot \partial_i\partial_j = \lambda \cdot D_1^2$$

for a constant $\lambda \in \mathbb{C}$. Bearing in mind the above considerations and the connection between F_X and the normal tangent cones $\mathcal{C}_{[J]}(\overline{\mathcal{J}}_{g+1}/\overline{\mathcal{J}}_g)$ for points $[J] \in \mathcal{J}_g$, one could consider a modified Γ_{00}-conjecture in the following terms:

Conjecture 3.2. (Modified Γ_{00}-conjecture)*: (X, Θ) is a Jacobian if and only if the connected component of F_X at the origin, F_X^0, has dimension at least one.*

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Abelian Solutions of the Soliton Equations and Geometry of Abelian Varieties

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Abstract. We introduce the notion of abelian solutions of the $2D$ Toda lattice equations and the bilinear discrete Hirota equation, and show that all of them are algebro-geometric.

Mathematics Subject Classification (2000). Primary 37K10, Secondary 14H70.

1. Introduction

The first goal of this paper is to extend the theory of abelian solutions of the Kadomtsev-Petviashvili (KP) equation developed recently in [23] to the case of the $2D$ Toda lattice

$$\partial_\xi \partial_\eta \varphi_n = e^{\varphi_{n-1}-\varphi_n} - e^{\varphi_n - \varphi_{n+1}}\,. \tag{1}$$

We call a solution $\varphi_n(\xi,\eta)$ of the equation *abelian* if it is of the form

$$\varphi_n(\xi,\eta) = \ln \frac{\tau((n+1)U+z,\xi,\eta)}{\tau(nU+z,\xi,\eta)}\,, \tag{2}$$

where $n \in \mathbb{Z}$, ξ, $\eta \in \mathbb{C}$ and $z \in \mathbb{C}^d$ are independent variables; for all ξ, η the function $\tau(\cdot,\xi,\eta)$ is a holomorphic section of a line bundle $\mathcal{L} = \mathcal{L}(\xi,\eta)$ on an abelian variety $X = \mathbb{C}^d/\Lambda$, i.e., it satisfies the monodromy relations

$$\tau(z+\lambda,\xi,\eta) = e^{a_\lambda \cdot z + b_\lambda}\tau(z,\xi,\eta)\,, \quad \lambda \in \Lambda\,,$$

for some $a_\lambda \in \mathbb{C}^d$, $b_\lambda = b_\lambda(\xi,\eta) \in \mathbb{C}$; and $U \in \mathbb{C}^d$, $U \notin \Lambda$.

A concept of abelian solutions of soliton equations provides an unifying framework for the theory of elliptic solutions of soliton equations and the theory of their rank 1 algebro-geometric solutions. The former corresponds to the case when the

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τ-function is a section of line bundle on an elliptic curve ($d = 1$), and the latter corresponds to the case when X is the Jacobian of an auxiliary algebraic curve and τ is the corresponding Riemann theta function.

Theory of elliptic solutions of the KP equation goes back to the work [1], where it was found that the dynamics of poles of an elliptic solution of the Korteweg-de Vries equation can be described in terms of the elliptic Calogero-Moser (CM) system with some conditions on the configuration of poles. In [14] it was shown that when the conditions are removed this correspondence becomes a full isomorphism between the solutions of the elliptic CM system and the elliptic solutions of the KP equation.

Elliptic solutions of the $2D$ Toda lattice were considered in [24] where it was shown that if $\tau(z, \xi, \eta)$ in (2) is an elliptic polynomial, i.e., if the τ-function of the $2D$ Toda lattice equation is of the form

$$\tau(z, \xi, \eta) = c(\xi, \eta) \prod_{i=1}^{N} \sigma(z - x_i(\xi, \eta)) \,,$$

then its zeros as functions of the variables ξ and η satisfy the equations of motion of the Ruijsenaars-Schneider (RS) system [27]:

$$\ddot{x}_i = \sum_{s \neq i} \dot{x}_i \dot{x}_s (V(x_i - x_s) - V(x_s - x_i)) \,, \quad V(x) = \zeta(x) - \zeta(x + \eta) \,,$$

which is a relativistic version of the elliptic CM system. Here and below $\sigma(z) = \sigma(z, 2\omega, 2\omega')$ and $\zeta(z) = \zeta(z, 2\omega, 2\omega')$ are the Weierstrass σ- and ζ-functions, respectively.

The correspondence between finite-dimensional integrable systems and pole systems of various soliton equations has been extensively studied in [4, 17, 18, 22] (see [5, 10, 19] and references therein for connections with the Hitchin-type systems).

A general scheme of constructing Lax representations with a spectral parameter, for systems using a specific inverse problem for linear equations with elliptic coefficients, is presented in [17]. Roughly speaking, this inverse problem is the problem of characterization of linear difference or differential equations with *elliptic* coefficients having solutions that are meromorphic sections of some line bundle on the corresponding elliptic curve (double-Bloch solutions).

Analogous problems for linear equations with coefficients that are meromorphic functions expressed in terms of the Riemann theta function of an indecomposable principally polarized abelian variety (ppav) X were a starting point in the recent proof in [20, 21] of Welters' remarkable trisecant conjecture: *an indecomposable ppav X is the Jacobian of a curve if and only if there exists a trisecant of its Kummer variety $K(X)$.*

Welters' conjecture, first formulated in [30], was motivated by Gunning's celebrated theorem [9] and by another famous conjecture: the Jacobians of curves are exactly the indecomposable ppavs whose theta-functions provide explicit solutions

of the KP equation. The latter was proposed earlier by Novikov and was unsettled at the time of Welters' work. It was proved later in [25].

Let B be an indecomposable symmetric matrix with positive definite imaginary part. It defines an indecomposable ppav $X = \mathbb{C}^g/\Lambda$, where the lattice Λ is generated by the basis vectors $e_m = (\delta_{m,i}) \in \mathbb{C}^g$ and the column-vectors B_m of B. The Riemann theta-function $\theta(z)$ corresponding to B is given by the formula

$$\theta(z) = \theta(z \,|\, B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \quad (z,m) = m_1 z_1 + \cdots + m_g z_g \,.$$

The Kummer variety $K(X)$ is an image of the Kummer map

$$K \colon X \ni Z \longmapsto (\Theta[\varepsilon_1, 0](Z) : \cdots : \Theta[\varepsilon_{2^g}, 0](Z)) \in \mathbb{CP}^{2^g-1},$$

where $\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z \,|\, 2B)$ are level two theta-functions with half-integer characteristics ε.

A trisecant of the Kummer variety is a projective line which meets $K(X)$ at least at three points. Fay's well-known trisecant formula [8] implies that if B is a matrix of b-periods of normalized holomorphic differentials on a smooth genus g algebraic curve Γ, then a set of three arbitrary distinct points on Γ defines a *one-parameter family* of trisecants parameterized by a fourth point of the curve. In [9] Gunning proved under certain non-degeneracy assumptions that the existence of such a family of trisecants characterizes Jacobian varieties among indecomposable ppavs.

Gunning's geometric characterization of the Jacobian locus was extended by Welters who proved that the Jacobian locus can be characterized by the existence of a formal one-parameter family of flexes of the Kummer varieties [29, 30]. A flex of the Kummer variety is a projective line which is tangent to $K(X)$ at some point up to order 2. It is a limiting case of trisecants when the three intersection points come together.

In [2] Arbarello and De Concini showed that Welters' characterization is equivalent to an infinite system of partial differential equations representing the KP hierarchy, and proved that only a finite number of these equations is sufficient. Novikov's conjecture that just the first equation of the hierarchy is sufficient for the characterization of the Jacobians is much stronger. It is equivalent to the statement that the Jacobians are characterized by the existence of length 3 formal jet of flexes.

Welters' conjecture that requires the existence of only one trisecant is the strongest. In fact, there are three particular cases of Welters' conjecture, which are independent and have to be considered separately. They correspond to three possible configurations of the intersection points (a, b, c) of $K(X)$ and the trisecant:

(i) all three points coincide;
(ii) two of them coincide;
(iii) all three intersection points are distinct.

In all of these cases the classical addition theorem for the Riemann theta-functions directly imply that secancy conditions are equivalent to the existence of certain

solutions for the auxiliary linear problems for the KP, the $2D$ Toda, and the bilinear discrete Hirota equations (BDHE), respectively.

For example, one of the Lax equations for the $2D$ Toda equation (1) is the differential-difference equation

$$\partial_t \psi_n(t) = \psi_{n+1}(t) - u_n(t)\psi_n(t) \tag{3}$$

with the potential u of the form

$$u_n(t) = \partial_t \ln \tau(n,t) - \partial_t \ln \tau(n+1,t)\,. \tag{4}$$

Let us assume that

$$\tau(n,t) = \theta(nU + tV + z)\,, \tag{5}$$

and equation (3) has a solution of the form

$$\psi_n(t) = \frac{\theta(A + nU + tV + z)}{\theta(nU + tV + z)}\, e^{np+tE}, \tag{6}$$

where p, E are constants and z is arbitrary. Then a direct substitution of (4), (5) and (6) into (3) gives the equation

$$E\theta(A+z)\theta(U+z) - e^p\theta(A+U+z)\theta(z) = \partial_V\theta(U+z)\theta(A+z) - \partial_V\theta(A+z)\theta(U+z)\,, \tag{7}$$

which is equivalent to the condition that the projective line passing through the points $\{K((A\pm U)/2)\}$ is tangent to the Kummer variety at the point $K((A-U)/2)$ (the case (ii) above).

The characterization of the Jacobian locus via (7) is the statement ([21]):

an indecomposable principally polarized abelian variety (X,θ) is the Jacobian of a smooth curve of genus g if and only if there exist non-zero g-dimensional vectors $U \neq A \pmod{\Lambda}$, V, such that equation (7) holds.

The "only if" part of the statement follows from the construction of solutions of the $2D$ Toda lattice equations in [15], from which it also follows that the vector A is a point of $\Gamma \subset J(\Gamma)$, the vector U is of the form $U = P_- - P_+$, where $P_\pm \in \Gamma$ are points on Γ (often called *punctures*, which define the $2D$ Toda flows), and the vector V is a tangent vector to Γ at one of the punctures.

In geometric terms the spectral curves of the elliptic RS system, that give elliptic solutions of (1), are singled out by the condition that there exist a pair of punctures $P_\pm$ such that the corresponding vector U *spans an elliptic curve* in $J(\Gamma)$.

For any curve Γ and any pair of points $P_\pm \in \Gamma$ the Zariski closure of the group $\{Un \mid n \in \mathbb{Z},\ U = P_- - P_+\}$ in $J(\Gamma)$ is an abelian subvariety $X \subset J(\Gamma)$. When X is a proper subvariety, i.e., $d := \dim X < g = \dim J(\Gamma)$, the restrictions of $\theta(tV + z)$ and $\theta(A + tV + z)$ on the corresponding linear subspace

$$\mathbb{C}^d := \big(\text{the component through the origin of } \pi^{-1}(X)\big) \subset \mathbb{C}^g,$$

where $\pi\colon \mathbb{C}^g \to J(\Gamma)$ is the covering map, can be seen as sections $\tau(z,t)$, $\tau_A(z,t)$ of some line bundles on X, i.e., they satisfy the monodromy properties with respect

to the lattice $\Lambda \subset \mathbb{C}^d$ such that $X = \mathbb{C}^d/\Lambda$:

$$\tau(z+\lambda,t) = e^{a_\lambda \cdot z + b_\lambda}\tau(z,t)\,,\quad \tau_A(z+\lambda,t) = e^{a_\lambda \cdot z + c_\lambda}\tau_A(z,t)\,,\quad \lambda \in \Lambda,\ z \in \mathbb{C}^d \tag{8}$$

for some $a_\lambda \in \mathbb{C}^d$, $b_\lambda = b_\lambda(t)$, $c_\lambda = c_\lambda(t) \in \mathbb{C}$.

Equation (7) restricted to $z \in \mathbb{C}^d$ takes the form

$$E\,\tau_A(z,t)\,\tau(U+z,t) - e^p\,\tau_A(U+z,t)\,\tau(z,t) = \dot\tau(z+U,t)\,\tau_A(z,t) - \tau(z+U,t)\,\dot\tau_A(z,t)\,. \tag{9}$$

Here and below "dot" stands for the derivative with respect to the variable t.

At first sight equation (9) considered as an equation for two unknown sections $\tau(z,t)$ and $\tau_A(z,t)$ of some line bundles $\mathcal{L}(t)$ and $\mathcal{L}_A(t)$ on an arbitrary abelian variety X is not as restrictive as finite-dimensional equation (7). Nevertheless, our first main result is that at least under certain genericity assumptions all the *abelian* solutions of equation (9) arise in the way described above, i.e., they are rank one algebro-geometric, and we have $X \subset J(\Gamma)$ for some algebraic curve Γ, which in general might be singular.

Theorem 1.1. *Suppose that for some p, $E \in \mathbb{C}$ and $0 \neq U \in \mathbb{C}^d$ equation* (9) *is satisfied by $\tau(z,t)$ and $\tau_A(z,t)$, which for all t are holomorphic functions in z satisfying the monodromy properties* (8)*. Assume, moreover, that*

(i) *Λ is maximal with this property, i.e., any $\lambda \in \mathbb{C}^d$ satisfying* (8) *for some $a_\lambda \in \mathbb{C}^d$ and $b_\lambda(t)$, $c_\lambda(t) \in \mathbb{C}$ must belong to Λ;*

(ii) *for each t the divisor $\mathcal{T}^t := \{z \in X \mid \tau(z,t) = 0\}$ is reduced and irreducible;*

(iii) *the group $\{Un \bmod \Lambda \mid n \in \mathbb{Z}\} \subset X$ is Zariski dense in X.*

Then there exist a unique irreducible algebraic curve Γ, smooth points $P_\pm \in \Gamma$, an injective homomorphism $j_0\colon X \to J(\Gamma)$ and a torsion-free rank 1 sheaf $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ of degree $g-1$, where $g = g(\Gamma)$ is the arithmetic genus of Γ, such that setting $j(z) = j_0(z) \otimes \mathcal{F}$ we have

$$\tau(Un+z,t) = \rho(t)\,\widehat\tau_n(t,0\,|\,\Gamma,P_\pm,j(z))\,,$$

where $\widehat\tau_n(t_1^+,t_1^-\,|\,\Gamma,P_\pm,\mathcal{F})$ is the 2D Toda τ-function defined by the data $(\Gamma,P_\pm,\mathcal{F})$.

Note that when Γ is smooth:

$$\widehat\tau_n(t_1^+,t_1^-\,|\,\Gamma,P,j(z)) = \theta\big(nU + t_1^+V_+ + t_1^-V_- + j(z)\ \big|\ B(\Gamma)\big)\,e^{Q(n,t_1^+,t_1^-)}\,,$$

where $V_\pm \in \mathbb{C}^d$, Q is a quadratic form, $B(\Gamma)$ is the matrix of B-periods of Γ, and θ is the Riemann theta function. Linearization in the Jacobian $J(\Gamma)$ of nonlinear t-dynamics for $\tau(z,t)$ provides some evidence that there might be underlying integrable systems on the spaces of higher level theta-functions on a ppav. The RS system is an example of such a system for $d = 1$.

Almost till the very end the proof of Theorem 1.1 goes along the lines of [21]. We would like to stress that the proof of the trisecant conjecture in [21] uses none of the assumptions above. We include assumption (iii) in the statement of the theorem only to avoid unnecessary analytical difficulties at this stage.

The second goal of this paper, discussed in the last section, is to study abelian solutions of the BDHE. The latter is a difference equation of the form

$$\tau_n(l+1,m)\tau_n(l,m+1)-\tau_n(l,m)\tau_n(l+1,m+1)+\tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1)=0\,.$$

One of its auxiliary Lax equations is the two-dimensional linear difference equation

$$\psi(m,n+1)=\psi(m+1,n)+u(m,n)\psi(m,n)$$

with the potential u of the form

$$u(m,n)=\frac{\tau(n+1,m+1)\,\tau(n,m)}{\tau(n+1,m)\,\tau(n,m+1)}\,.$$

Under the light-cone change of variables

$$x=m-n,\quad \nu=m+n\,, \tag{10}$$

and under the assumption that $\tau(n,m)$ is of the form $\tau(Wx+z,\nu)$ with $z,W\in\mathbb{C}^d$, equation (3) gets transformed to the difference-functional equation

$$\psi(z-W,\nu)=\psi(z+W,\nu)+u\psi(z,\nu-1)\,, \tag{11}$$

with

$$u(z,\nu)=\frac{\tau(z,\nu+1)\,\tau(z,\nu-1)}{\tau(z-W,\nu)\,\tau(z+W,\nu)}\,. \tag{12}$$

Equation (11) for ψ of the form

$$\psi(x,\nu)=\frac{\tau_A(z,\nu)}{\tau(z,\nu)}\,e^{p\cdot z+\nu E}$$

is equivalent to the discrete analog of (9)

$$\begin{aligned}e^{-p\cdot W}\tau(z+W,\nu)\tau_A(z-W,\nu)&\\ =e^{p\cdot W}\tau(z-W,\nu)\tau_A(z+W,\nu)+e^{-E}\tau(z,\nu+1)\tau_A(z,\nu-1)\,,&\end{aligned} \tag{13}$$

where, as before, $\tau(z,\nu)$ and $\tau_A(z,\nu)$ are sections of some line bundles on X, i.e., they are holomorphic functions satisfying the monodromy properties

$$\tau(z+\lambda,\nu)=e^{a_\lambda\cdot z+b_\lambda(\nu)}\tau(z,\nu)\,,\quad \tau_A(z+\lambda,\nu)=e^{a_\lambda\cdot z+c_\lambda(\nu)}\tau_A(z,\nu)\,,\quad \lambda\in\Lambda, \tag{14}$$

with respect to the lattice Λ of an abelian variety $X=\mathbb{C}^d/\Lambda$. If X is ppav and $\tau(z,\nu)=\theta(z+V\nu)$, $\tau_A(z,\nu)=\theta(A+z+V\nu)$, then (13) is equivalent to the trisecant equation

$$e^{-p\cdot W}\theta(z+W)\theta(z+A-W)=e^{p\cdot W}\theta(z+A+W)\theta(z-W)+e^{-E}\theta(z+V)\theta(z+A-V)\,.$$

We conjecture that under the assumption that $\tau(z,\nu)$, $\tau_A(z,\nu)$ are *meromorphic quasiperiodic functions of the variable* ν all the *abelian* solutions of equation (13) are rank one algebro-geometric, and we have $X\subset J(\Gamma)$ for some algebraic curve Γ, (which in general might be singular). The main result of the last section is a proof of this conjecture in the case when $\tau(z,\nu)$ is periodic in the variable ν with some sufficiently large prime period N. More precisely,

Theorem 1.2. *Suppose that the equation* (13) *with some* $p, E \in \mathbb{C}$ *and* $0 \neq W \in \mathbb{C}^d$, *is satisfied with* $\tau(z,\nu)$, $\tau_A(z,\nu)$, *such that for all* ν *the functions* $\tau_A(z,\nu)$ *and* $\tau(z,\nu)$ *are holomorphic functions satisfying the monodromy properties* (14) *with respect to the lattice* Λ *of an abelian variety* $X = \mathbb{C}^d/\Lambda$. *Assume, moreover, that*

(i) Λ *is maximal with this property, i.e., any* $\lambda \in \mathbb{C}^d$ *satisfying* (14) *for some* $a_\lambda \in \mathbb{C}^d$ *and* $b_\lambda(\nu)$, $c_\lambda(\nu) \in \mathbb{C}$ *must belong to* Λ, *and that*

(ii) *for each* ν *the divisor* $\mathcal{T}^\nu := \{z \in X \mid \tau(z,\nu) = 0\}$ *is reduced and irreducible;*

(iii) *the Zariski closure of the group* $\{2Wm \bmod \Lambda \mid m \in \mathbb{Z}\}$ *in* X *coincides with* X*;*

(iv) *the functions* $\tau(z,\nu)$, $\tau_A(z,\nu)$ *are meromorphic functions of the variable* $\nu \in \mathbb{C}$ *and* $\tau(z,\nu)$ *is a quasiperiodic function of* ν, *satisfying the monodromy relation*

$$\tau(z,\nu+N) = e^{a\cdot z + c\,\nu}\tau(z,\nu) \tag{15}$$

with an integer prime period $N > \dim H^0(\mathcal{T}^\nu)$ *and with some* $a \in \mathbb{C}^d$, $c \in \mathbb{C}$.

Then there exist a unique irreducible algebraic curve Γ, *smooth points* P_0, P_1, $P_2 \in \Gamma$, *an injective homomorphism* $j_0 \colon X \to J(\Gamma)$ *and a torsion-free rank 1 sheaf* $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ *of degree* $g-1$, *where* $g = g(\Gamma)$ *is the arithmetic genus of* Γ, *such that setting* $j(z) = j_0(z) \otimes \mathcal{F}$ *we have*

$$\tau(Wx+z,\nu) = \rho(\nu)\,\widehat{\tau}(x,\nu,0,\dots \mid \Gamma, P_i, j(z))\,,$$

where $\widehat{\tau}(t_1,t_2,t_3,\dots \mid \Gamma, P_i, \mathcal{F})$ *is the BDHE* τ*-function defined by the data* $(\Gamma, P_i, \mathcal{F})$.

2. Construction of the wave function

Equation (9) is equivalent to equation (3) with

$$u_n = -\partial_t \ln \frac{\tau((n+1)U+z,t)}{\tau(nU+z,t)}, \quad \psi_n = \frac{\tau_A(nU+z,t)}{\tau(nU+z,t)} e^{P\cdot z + Et}, \tag{16}$$

where $P \in \mathbb{C}^d$ is a vector such that $P \cdot U = p$. In the core of the proof of Theorem is the construction of quasiperiodic wave function as in (23), (24) below, which contains much more information than the function ψ in (16) having no spectral parameter. We would like to emphasize once again that the construction of wave function follows closely the argument from the beginning of Section 2 in [21] but is drastically simplified by assumption (ii) in the statement of the theorem.

The construction is presented in two steps. First we show that the existence of a holomorphic solutions of equation (9) implies certain relations on the tau divisor $\mathcal{T}^t$.

Lemma 2.1. *If equation* (9) *has holomorphic solutions whose divisors have no common components* (*or if the* τ*-divisor is irreducible*), *then the equation*

$$\partial_t^2\tau(z,t)\,\tau(z+U,t)\,\tau(z-U,t) = \partial_t\tau(z,t)\,\partial_t\left(\tau(z+U,t)\,\tau(z-U,t)\right) \tag{17}$$

is valid on the divisor $\mathcal{T}^t = \{z \in \mathbb{C}^d \mid \tau(z,t) = 0\}$.

In [21] equation (17) was derived with the help of pure local consideration. Let us show that they can be easily obtained globally.

Proof. The evaluations of (9) on the divisors $\mathcal{T}^t$ and $\mathcal{T}^t - U$ give

$$(\dot\tau_A(z) + E\tau_A(z))\tau(z+U) = \dot\tau(z+U)\tau_A(z), \quad z \in \mathcal{T}^t, \tag{18}$$

$$\tau_A(z)\tau(z-U) + \dot\tau(z)\tau_A(z-U)e^{-p} = 0, \quad z \in \mathcal{T}^t\,. \tag{19}$$

Here and below for brevity we omit the notations for explicit dependence of functions on the variable t, thus $\tau(z) = \tau(z,t)$, $\tau_A(z) = \tau_A(z,t)$.

Evaluating the derivative of (9) on $\mathcal{T}^t - U$ gives another equation

$$(E\tau_A(z) + \dot\tau_A(z))\tau(z-U) + \dot\tau(z-U)\tau_A(z) + \ddot\tau(z)\tau_A(z-U)e^{p} = 0, \quad z \in \mathcal{T}^t\,. \tag{20}$$

Eliminating $\tau_A(z-U)$ and $\dot\tau_A(z)$ from (18)–(20) we obtain the equation

$$[\ddot\tau(z)\,\tau(z+U)\,\tau(z-U) - \dot\tau(z,t)\,\partial_t\,(\tau(z+U)\,\tau(z-U))]\,\tau_A(z) = 0, \quad z \in \mathcal{T}^t,$$

which implies (17) due to the assumption that the divisors of τ and τ_A have no common components (or under the assumption that $\mathcal{T}^t$ is irreducible).

In [21] it was shown that equation (17) is sufficient for the existence of *local* meromorphic wave solutions of (3) which are holomorphic outside the zeros of τ. Let us show that in a global setting they are sufficient for the existence of quasi-periodic wave solutions of the differential-functional equation:

$$\partial_t\psi(z,t) = \psi(z+U,t) - u(z,t)\psi(z,t) \tag{21}$$

with

$$u = \partial_t \ln\tau(z,t) - \partial_t \ln\tau(z+U,t)\,, \tag{22}$$

which restricted to the points $z + Un$ takes the form (3).

The wave solution of (21) is a formal solution of the form

$$\psi = k^{l\cdot z}e^{kt}\phi(z,t,k)\,, \tag{23}$$

where l is a vector $l \in \mathbb{C}^d$ such that $l \cdot U = 1$ and ϕ is a formal series

$$\phi(z,t,k) = e^{bt}\left(1 + \sum_{s=1}^{\infty}\xi_s(z,t)\,k^{-s}\right). \tag{24}$$

Lemma 2.2. *Let equation* (17) *for* $\tau(z,t)$ *hold, and let* $\lambda_1,\dots,\lambda_d$ *be a set of* $\mathbb{C}$*-linearly independent vectors in the lattice* Λ*. Then equation* (21) *with* u *as in* (22) *has a unique, up to a* z*-independent factor, wave solution such that:*

(i) *the coefficients* $\xi_s(z,t)$ *of the formal series* (24) *are meromorphic functions of the variable* $z \in \mathbb{C}^d$ *with a simple pole at the divisor* $\mathcal{T}^t$*, i.e.,*

$$\xi_s(z,t) = \frac{\tau_s(z,t)}{\tau(z,t)}\,,$$

and $\tau_s(z,t)$ *is a holomorphic function of* z*;*

(ii) *$\phi(z,t,k)$ is quasi-periodic with respect to the lattice Λ:*

$$\phi(z+\lambda,t,k)=\phi(z,t,k)\,B^{\lambda}(k),\quad \lambda\in\Lambda\,, \tag{25}$$

and is periodic with respect to the vectors $\lambda_1,\dots,\lambda_d$, i.e.,

$$B^{\lambda_i}(k)=1,\quad i=1,\dots,d\,. \tag{26}$$

Proof. The functions $\xi_s(z)$ are defined recursively by the equations

$$\Delta_U\,\xi_{s+1}=\dot{\xi}_s+(u+b)\,\xi_s\,. \tag{27}$$

Here and below Δ_U stands for the difference derivative

$$\Delta_U=e^{\partial_U}-1.$$

The quasi-periodicity conditions (25) for ϕ are equivalent to the equations

$$\xi_s(z+\lambda,t)-\xi_s(z,t)=\sum_{i=1}^{s}B_i^{\lambda}\xi_{s-i}(z,t)\,,\quad \xi_0=1\,. \tag{28}$$

The general quasi-periodic solution of the first equation $\Delta_U\,\xi_1=u+b$ is given by the formula

$$\xi_1=-\partial_t\ln\tau+l_1(z,t)\,b+c_1(t),$$

where $l_1(z,t)$ is a linear form on $\mathbb{C}^d$ such that $l_1(U,t)=1$. It satisfies the monodromy relations (28) with

$$B_1^{\lambda}=l_1(\lambda)\,b-\partial_t\ln\tau(z+\lambda,t)+\partial_t\ln\tau(z,t)=l_1(\lambda,t)\,b-\dot{b}_\lambda(t)\,,$$

where $b_\lambda=b_\lambda(t)$ are defined in (8). The normalizing conditions $\mathcal{B}_1^{\lambda_i}=0$, $i=1,\dots,d$ uniquely define the constant b and the linear form $l_1(z)$.

Let us assume that the coefficient ξ_{s-1} of the series (24) is known, and that there exists a solution ξ_s^0 of the next equation, which is holomorphic outside of the divisor $\mathcal{T}^t$, and which satisfies the quasi-periodicity conditions (28) with $B_s^{\lambda_j}=0$ and possibly t-dependent coefficient $B_s^{\lambda}(t)$, for $\lambda\neq\lambda_j$, i.e.,

$$\xi_s(z+\lambda,t)-\xi_s(z,t)=B_s^{\lambda}(t)+\sum_{i=1}^{s-1}B_i^{\lambda}\xi_{s-i}(z,t),\quad B_s^{\lambda_j}=0\,.$$

We assume also that ξ_s^0 is unique up to the transformation $\xi_s=\xi_s^0+c_s(t)$, where $c_s(t)$ is a time-dependent constant.

Let us define a function $\tau_{s+1}^0(z)$ on $\mathcal{T}^t$ with the help of the formula

$$\tau_{s+1}^0=-\partial_t\tau_s(z,t)-b\tau_s(z,t)+\frac{\partial_t\tau(z+U,t)}{\tau(z+U,t)}\tau_s(z,t),\quad z\in\mathcal{T}^t. \tag{29}$$

Let us show that the formula (29) can be written also in the alternative form:

$$\tau_{s+1}^0=-\partial_t\tau(z,t)\frac{\tau_s(z-U,t)}{\tau(z-U,t)},\quad z\in\mathcal{T}^t. \tag{30}$$

By the induction assumption, $\xi_s = (\tau_s/\tau)$ is a solution of (27) for $s-1$, i.e., the function τ_s satisfies the equation

$$[\dot\tau_{s-1}(z-U)+\tau_s(z-U)+b\tau_{s-1}(z-U)]\,\tau(z) = \tau_s(z)\tau(z-U)+\dot\tau(z)\,\tau_{s-1}(z-U), \quad (31)$$

where once again we omit notations for explicit dependence of all the functions on the variable t.

From (31) it follows that

$$\tau_s(z)\tau(z-U) + \dot\tau(z)\,\tau_{s-1}(z-U) = 0, \quad z \in \mathcal{T}^t\,. \quad (32)$$

The evaluation of the derivative of (31) at $\mathcal{T}^t$ implies

$$(\tau_s(z-U)+b\tau_{s-1}(z-U))\,\dot\tau(z) = \dot\tau_s(z)\,\tau(z-U)+\tau_s(z)\,\dot\tau(z-U)+\ddot\tau(z)\tau_{s-1}(z-U)\,, \quad z \in \mathcal{T}^t.$$

Then, using (17) and (32) we obtain the equation

$$\frac{\dot\tau(z)\tau_s(z-U)}{\tau(z-U)} = b\tau_s(z) + \dot\tau_s(z) - \frac{\dot\tau(z+U)\tau_s(z)}{\tau(z+U)}\,. \quad (33)$$

Hence the expressions (29) and (30) do coincide.

The expression (29) is certainly holomorphic when $\tau(z+U)$ is non-zero, i.e., is holomorphic outside of $\mathcal{T}^t \cap (\mathcal{T}^t - U)$. Similarly from (30) we see that $\tau^0_{s+1}(z,t)$ is holomorphic away from $\mathcal{T}^t \cap (\mathcal{T}^t + U)$.

We claim that $\tau^0_{s+1}(z,t)$ is holomorphic everywhere on $\mathcal{T}^t$. Indeed, by assumption (iii) in Theorem 1.1 the abelian subgroup generated by U is Zariski dense in X. Therefore, for any point $z_0 \in \mathcal{T}^t$ there exists an integer $k \geq 0$ such that $z_k = z_0 - kU$ is in $\mathcal{T}^t$, and $\tau(z_{k+1}, t) \neq 0$. Then, from equation (30) it follows that τ^0_{s+1} is regular at the point $z = z_k$. Using equation (29) for $z = z_k$, we get that $\partial_t\tau(z_{k-1},t)\tau_s(z_k,t) = 0$. The last equality and the equation (30) for $z = z_{k-1}$ imply that τ^0_{s+1} is regular at the point z_{k-1}. Regularity of τ^0_{s+1} at z_{k-1} and equation (29) for $z = z_{k-1}$ imply $\partial_t\tau(z_{k-2},t)\tau_s(z_{k-1},t) = 0$. Then equation (30) for $z = z_{k-2}$ implies that τ^0_{s+1} is regular at the point z_{k-2}. By continuing these steps we get finally that τ^0_{s+1} is regular at $z = z_0$. Therefore, τ^0_{s+1} is regular on $\mathcal{T}^t$.

Recall, that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension onto $\mathbb{C}^d$ ([28]). Therefore, there exists a holomorphic function $\tilde\tau(z,t)$ such that $\tilde\tau_{s+1}|_{\mathcal{T}^t} = \tau^0_{s+1}$. Consider the function $\chi_{s+1} = \tilde\tau_{s+1}/\tau$. It is holomorphic outside of the divisor $\mathcal{T}^t$. From (28) and (30) it follows that the function $f^\lambda_{s+1}(z)$ defined by the equation

$$\chi_{s+1}(z+\lambda) - \chi_{s+1}(z) = f^\lambda_{s+1}(z) + \sum_{i=1}^{s} B_i{}^\lambda \xi_{s+1-i}(z)\,,$$

has no pole at $\mathcal{T}^t$, i.e., it is a holomorphic function of $z \in \mathbb{C}^d$. It satisfies the twisted homomorphism relations

$$f^{\lambda+\mu}_{s+1}(z) = f^\lambda_{s+1}(z+\mu) + f^\mu_{s+1}(z)\,,$$

i.e., it defines an element of the first cohomology group of Λ with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda, H^0(\mathbb{C}^d, \mathcal{O}))$. The same arguments, as that used in the proof of the part (b) of Lemma 12 in [25], show that there exists a holomorphic function $h_{s+1}(z)$ such that

$$f^\lambda_{s+1}(z) = h_{s+1}(z+\lambda) - h_{s+1}(z) + \widetilde{B}^\lambda_{s+1},$$

where $\widetilde{B}^\lambda_{s+1} = \widetilde{B}^\lambda_{s+1}(t)$ is a time-dependent constant. Hence the function $\zeta_{s+1} = \chi_{s+1} + h_{s+1}$ has the following monodromy properties

$$\zeta_{s+1}(z+\lambda) - \zeta_{s+1}(z) = \widetilde{B}^\lambda_{s+1} + \sum_{i=1}^{s} B^\lambda_i \xi_{s+1-i}(z)\,, \tag{34}$$

Let us consider the function

$$R_{s+1} = \zeta_{s+1}(z+U) - \zeta_{s+1}(z) - \dot{\xi}_s(z) - (u(z)+b)\,\xi_s(z)\,.$$

From equation (29), (30) it follows that it has not poles at $\mathcal{T}^t$ and $\mathcal{T}^t - U$, respectively. Hence $R_{s+1}(z)$ is a holomorphic function.

From (34) it follows that it satisfies the following monodromy properties

$$R_{s+1}(z+\lambda) = R_{s+1}(z) - \dot{B}^\lambda_s\,.$$

Recall, that by the induction assumption $B^{\lambda_j}_s = 0$, where λ_j, $j = 1, \ldots, d$, are linear independent. Therefore, R_{s+1} is a constant (z-independent) and B^λ_s for all λ are in fact t-independent.

The function

$$\widetilde{\xi}_{s+1}(z,t) = \zeta_{s+1}(z,t) + l_{s+1}(z,t) + c_{s+1}(t)\,,$$

where l_{s+1} is a linear form such that

$$l_{s+1}(U,t) = -R_{s+1}(t)\,,$$

is a solution of (27).

Under the transformation $\xi_s \longmapsto \xi_s(z,t) + c_s(t)$ which does not change the monodromy properties of ξ_s, the solution $\widetilde{\xi}_{s+1}$ gets transformed to

$$\xi_{s+1} = \widetilde{\xi}_{s+1} + \dot{c}_s(t) l_1(z,t) + c_s(t)\xi_1(z,t)\,,$$

where $l_1(z,t)$ is the linear form defined above in the initial step of the induction. The new solution ξ_{s+1} satisfies the monodromy relations (28) with constant B^λ_i for $i \le s$ and with t-dependent coefficient

$$B^\lambda_{s+1}(t) = \widetilde{B}^\lambda_{s+1}(t) + l_{s+1}(\lambda,t) + \dot{c}_s(t) l_1(\lambda,t) + c_s(t) B^\lambda_1\,.$$

The normalization condition (26) for $B^{\lambda_i}_{s+1} = 1, i = 0, \ldots, d$ defines uniquely l_{s+1} and $\partial_t c_s$, i.e., the time-dependence of $c_s(t)$. The induction step is completed.

Note that the remaining ambiguity in the definition of ξ_s on each step is the choice of a time-independent constant c_s. That corresponds to the multiplication of ψ by a constant formal series and thus the lemma is proven.

3. Commuting difference operators

Our next goal is to construct rings $\mathcal{A}^z$ of commuting difference operators parameterized by points $z \in X$. In fact the construction of such operators completes the proof of Theorem 1.1 because as shown in [26, 13] there is a natural correspondence

$$\mathcal{A} \longleftrightarrow (\Gamma, P_\pm, \mathcal{F}) \tag{35}$$

between commutative rings $\mathcal{A}$ of ordinary linear difference operators containing a pair of monic operators of co-prime orders, and sets of algebro-geometric data $(\Gamma, P_\pm, [k^{-1}]_1, \mathcal{F})$, where Γ is an algebraic curve with a fixed first jet $[k^{-1}]_1$ of a local coordinate k^{-1} in the neighborhood of a smooth point $P_+ \in \Gamma$ and $\mathcal{F}$ is a torsion-free rank 1 sheaf on Γ such that

$$h^0(\Gamma, \mathcal{F}(nP_+ - nP_-)) = h^1(\Gamma, \mathcal{F}(nP_+ - nP_-)) = 0\,. \tag{36}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo conjugation $\mathcal{A}' = g(x)\mathcal{A}g^{-1}(x)$.

The construction of the correspondence (35) depends on a choice of initial point $x_0 = 0$. The spectral curve and the sheaf $\mathcal{F}$ are defined by the evaluations of the coefficients of generators of $\mathcal{A}$ at a finite number of points of the form $x_0 + n$. In fact, the spectral curve is independent on the choice of x_0, but the sheaf does depend on it, i.e., $\mathcal{F} = \mathcal{F}_{x_0}$.

Using the shift of the initial point it is easy to show that the correspondence (35) extends to the commutative rings of operators whose coefficients are *meromorphic* functions of x. The rings of operators having poles at $x = 0$ correspond to sheaves for which the condition (36) for $n = 0$ is violated.

The algebraic curve Γ is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_+, P_-)$ of meromorphic functions on Γ with the only pole at the point P_+ and which vanish at P_-. The isomorphism is defined by the equation

$$L_a\psi_0 = a\psi_0, \quad L_a \in \mathcal{A},\ a \in A(\Gamma, P_+, P_-)\,.$$

Here ψ_0 is a common eigenfunction of the commuting operators. At $x = 0$ it is a section of the sheaf $\mathcal{F} \otimes \mathcal{O}(P_+)$.

Let

$$T = e^{\partial_U}.$$

In order to construct rings of commutative operators we first introduce a unique pseudo-difference operator

$$\mathcal{L}(z,t) = T + \sum_{s=0}^{\infty} w_s(z,t)\, T^{-s},$$

such that the equation

$$\mathcal{L}(z,t)\psi(z,t) = k\psi(z,t)\,, \tag{37}$$

with ψ given by (23), holds. The coefficients $w_s(z,t)$ of $\mathcal{L}$ are difference polynomials of the coefficients ξ_s of ϕ. Due to the quasiperiodicity of ψ they are meromorphic functions on the abelian variety X.

Let $\mathcal{L}^m_+$ be the strictly positive difference part of the operator $\mathcal{L}^m$, so that

$$\mathcal{L}^m_- = \mathcal{L}^m - \mathcal{L}^m_+ = F_m + \sum_{s=1}^{\infty} F^s_m T^{-s}.$$

By definition the leading coefficient F_m of $\mathcal{L}^m_-$ is the *residue* of $\mathcal{L}^m$:

$$F_m = \operatorname{res}_T \mathcal{L}^m,\ F^1_m = \operatorname{res}_T \mathcal{L}^m T.$$

From (21) and (37) it follows that $[\partial_t - T + u, \mathcal{L}^n] = 0$. Hence

$$[\partial_t - T + u, \mathcal{L}^m_+] = -[\partial_t - T + u, \mathcal{L}^m_-] = (\Delta_U F_m)\, T. \tag{38}$$

Indeed, the left-hand side of (38) is a difference operator with non-vanishing coefficients only at the positive powers of T, while the second member of (38) is at most of order 1. Therefore, it has the form $f_m T$. The coefficient f_m is easily expressed in terms of the leading coefficient of $\mathcal{L}^m_-$. Note that the vanishing of the coefficients of T^0 and T^{-1} implies the equation

$$\Delta_U F^1_m = \partial_t F_m\,, \tag{39}$$

$$\Delta_U F^2_m = \partial_t F^1_m + uF_1 - F_1(T^{-1}u)\,, \tag{40}$$

which we will use later.

The functions $F_m(z)$ are difference polynomials in the coefficients w_s of $\mathcal{L}$. Hence $F_m(z)$ are meromorphic functions on X.

Lemma 3.1. *There exist holomorphic functions $q_m(z,t)$ such that the equality*

$$F_m = \frac{q_m(z+U,t)}{\tau(z+U,t)} - \frac{q_m(z,t)}{\tau(z,t)} \tag{41}$$

holds.

Proof. If ψ is the wave solution as in (21)–(24), then there exists a unique pseudo-difference operator Φ such that

$$\psi = \Phi k^{P\cdot z} e^{kt},\quad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(s,t) T^{-s}.$$

The coefficients φ_s of Φ are universal difference polynomials of the coefficients ξ_s of ϕ. Therefore, $\varphi_s(z,t)$ is a meromorphic function of z. Note that (37) implies $\mathcal{L} = \Phi T \Phi^{-1}$.

The right action of a pseudo-difference operator is the formal adjoint action under which T acts on a function f as the opposite shift: $(fT) = T^{-1}f$. Consider the dual wave function defined by the right action of the operator Φ^{-1}:

$$\psi^+ = \left(k^{-P\cdot z} e^{-kt}\right) \Phi^{-1}.$$

If ψ is a formal wave solution of (21), then ψ^+ is a solution of the adjoint equation

$$(-\partial_t - T^{-1} + u)\, \psi^+ = 0\,.$$

If ψ is as in Lemma 2.2, then the dual wave solution is of the form

$$\psi^+ = k^{-P\cdot z} e^{-kt} \phi^+(Ux + z, t, k),$$

where, as before, due to (17) the coefficients $\xi_s^+(z,t)$ of

$$\phi^+(z,t,k) = e^{-bt}\left(1 + \sum_{s=1}^{\infty} \xi_s^+(z,t)\, k^{-s}\right)$$

have simple poles along the divisor $\mathcal{T}^t - U$.

The ambiguity in the definition of ψ does not affect the product

$$\psi^+\psi = \left(k^{-x}e^{-kt}\Phi^{-1}\right)\left(\Phi k^x e^{kt}\right). \tag{42}$$

Therefore, the coefficients J_s of the product

$$\psi^+\psi = \phi^+(z,t,k)\,\phi(z,t,k) = 1 + \sum_{s=1}^{\infty} J_s(z,t)\, k^{-s} \tag{43}$$

are meromorphic functions on X. The factors in the left-hand side of (43) have the simple poles on $\mathcal{T}^t$ and $\mathcal{T}^t - U$. Hence $J_s(z)$ is a meromorphic function on X with the simple poles at $\mathcal{T}^t$ and $\mathcal{T}^t - U$. Moreover, the left and right actions of pseudo-difference operators are formally adjoint, i.e., for any two operators $\mathcal{D}_1$, $\mathcal{D}_2$ the equality $\left(k^{-x}\mathcal{D}_1\right)\left(\mathcal{D}_2 k^x\right) = k^{-x}\left(\mathcal{D}_1\mathcal{D}_2 k^x\right) + (T-1)\left(k^{-x}\left(\mathcal{D}_3 k^x\right)\right)$ holds for some pseudo-difference operator $\mathcal{D}_3$ whose coefficients are difference polynomials in the coefficients of $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, from (42) it follows that

$$\psi^+\psi = 1 + \sum_{s=1}^{\infty} J_s k^{-s} = 1 + \Delta_U\left(\sum_{s=2}^{\infty} Q_s k^{-s}\right). \tag{44}$$

The coefficients of the series Q are difference polynomials in the coefficients φ_s of the wave operator Φ. Therefore, they are meromorphic functions of z with poles on $\mathcal{T}^t$, i.e., $Q_s = q_s/\tau$.

From the definition of $\mathcal{L}$ it follows that

$$\operatorname{res}_k\left(\psi^+(\mathcal{L}^n\psi)\right)k^{-1}dk = \operatorname{res}_k\left(\psi^+ k^n\psi\right)k^{-1}dk = J_n\,.$$

On the other hand, using the identity

$$\operatorname{res}_k\left(k^{-x}\mathcal{D}_1\right)\left(\mathcal{D}_2 k^x\right)k^{-1}dk = \operatorname{res}_T\left(\mathcal{D}_2\mathcal{D}_1\right), \tag{45}$$

we get

$$\operatorname{res}_k(\psi^+\mathcal{L}^n\psi)k^{-1}dk = \operatorname{res}_k\left(k^{-x}\Phi^{-1}\right)\left(\mathcal{L}^n\Phi k^x\right)k^{-1}dk = \operatorname{res}_T\mathcal{L}^n = F_n\,.$$

Therefore, $F_n = J_n$ and the lemma is proved.

Important remark. In [21] the statement that F_m has poles only along $\mathcal{T}^t$ and $\mathcal{T}^t - U$ was crucial for the proof of the existence of commuting difference operators associated with u. Namely, it implies that for all but a finite number of positive integers $i \notin A$ there exist constants $c_{n,\alpha}$ such that

$$F_i(z,t) - \sum_{\alpha\in A} c_{i,\alpha}F_\alpha(z,t) = 0\,,$$

hence (38) would imply that the corresponding linear combinations $L_i := \mathcal{L}_+^i - \sum c_{i,\alpha}\mathcal{L}_+^\alpha$ commutes with $P := \partial_t - T - u$. Not so: since these constants $c_{i,\alpha}$ might

depend on t, we might not have $[P, L_n] = 0$, and we cannot immediately make the next step and claim the existence of commuting operators (!).

So our next goal is to show that these constants in fact are t-independent. For that let us consider the functions $F_i^1(z,t)$. From (39) and (41) it follows that

$$F_i^1 = \partial_t \left(\frac{q_i(z,t)}{\tau(z,t)} \right). \tag{46}$$

Let $\{F_\alpha^1 \mid \alpha \in A\}$, for finite set A, be a basis of the space $\mathcal{F}(t)$ spanned by $\{F_m^1\}$. Then for all $n \notin A$ there exist constants $c_{n,\alpha}(t)$ such that

$$F_n^1(z,t) = \sum_{\alpha \in A} c_{n,\alpha}(t) F_\alpha^1(z,t)\,. \tag{47}$$

Due to (46) it is equivalent to the equations

$$q_n(z,t) = \sum_\alpha c_{n,\alpha}(t) q_\alpha(z,t)\,, \quad z \in \mathcal{T}^t,$$
$$\dot{q}_n^1(z,t) = \sum_\alpha c_{n,\alpha}(t) \dot{q}_\alpha^1(z,t)\,, \quad z \in \mathcal{T}^t\,,$$

from which we get

$$\sum_\alpha (\dot{c}_{n,\alpha}) q_\alpha(z,t) = 0\,, \quad z \in \mathcal{T}^t.$$

From (40) we obtain

$$\Delta_U \left(F_n^2 - \sum_{\alpha \in A} c_{n,\alpha}(t) F_\alpha^2(z,t) \right) = \dot{c}_{n,\alpha} F_\alpha^1\,.$$

The left-hand side is Δ_U derivative of a meromorphic function. The right-hand side has pole only at $\mathcal{T}^t$. Therefore, both sides of the equation must vanish. Then the assumption that the set F_α^1 is minimal imply $\dot{c}_{n,\alpha} = 0$.

Lemma 3.2. *Let ψ be a wave function corresponding to u, and let L_i, $i \notin A$ be the difference operator given by the formula*

$$L_i = \mathcal{L}_+^i - \sum_{\alpha \in A} c_{i,\alpha} \mathcal{L}_+^\alpha,\ i \notin A\,,$$

where the constants $c_{i,\alpha}$ are defined by equations (47).

Then the equation

$$L_i\,\psi = a_i(k)\,\psi, \quad a_i(k) = k^i + \sum_{s=1}^{\infty} a_{s,i} k^{n-s}\,, \tag{48}$$

where $a_{s,i}$ are constants, hold.

Proof. First note that from (38) it follows that

$$[\partial_t - T - u, L_i] = 0\,.$$

Hence, if ψ is the wave solution of (3), then $L_i\psi$ is also a wave solution of the same equation. By uniqueness of the wave function up to a constant in z-factor we get (48) and thus the lemma is proven.

The operator L_i can be regarded as a z-parametric family of ordinary difference operators L_i^z.

Corollary 3.1. *The operators L_i^z commute with each other,*

$$[L_i^z, L_j^z] = 0\,. \tag{49}$$

From (48) it follows that $[L_i^z, L_j^z]\psi = 0$. The commutator is an ordinary difference operator. Hence the last equation implies (49).

4. The fully discrete case

The main goal of this section is to characterize under some nondegeneracy assumptions all the abelian solutions of equation (13). As above we begin with the construction of the corresponding quasiperiodic wave function. We would like to emphasize once again that the construction of wave function follows closely the argument from the beginning of Section 5 in [21] but is simplified by the assumption (*iii*) in the formulation of Theorem 1.2.

4.1. Construction of the wave function

First let us show that the existence of a holomorphic solutions of equation (13) implies certain relations on $\mathcal{T}^\nu$.

Lemma 4.1 ([21])**.** *If equation* (13) *has holomorphic solutions, then the equation*

$$\frac{\tau(z+W,\nu+1)\,\tau(z-2W,\nu)\,\tau(z+W,\nu-1)}{\tau(z-W,\nu+1)\,\tau(z+2W,\nu)\,\tau(z-W,\nu-1)} = -1 \tag{50}$$

is valid on the divisor $\mathcal{T}^\nu = \{\, z \in \mathbb{C}^d \mid \tau(z,\nu) = 0\}$.

Proof. The evaluations of (13) at the divisors $\mathcal{T}^\nu \pm W$ give two different expressions for the restriction of $\tau_A(z,\nu)$ on $\mathcal{T}^\nu$:

$$\tau_A(z,\nu) = e^{p\cdot W - E}\,\frac{\tau(z+W,\nu+1)\,\tau_A(z+W,\nu-1)}{\tau(z+2W,\nu)}\,, \quad z \in \mathcal{T}^\nu, \tag{51}$$

$$\tau_A(z,\nu) = -e^{-p\cdot W - E}\,\frac{\tau(z-W,\nu+1)\,\tau_A(z-W,\nu-1)}{\tau(z-2W,\nu)}\,, \quad z \in \mathcal{T}^\nu\,. \tag{52}$$

The evaluation of equation (13) for $\nu - 1$ at $\mathcal{T}^\nu$ implies

$$e^{-p\cdot W}\tau(z+W,\nu-1)\,\tau_A(z-W,\nu-1) = e^{p\cdot W}\tau(z-W,\nu-1)\tau_A(z+W,\nu-1), \quad z \in \mathcal{T}^\nu. \tag{53}$$

Taking the ratio of (51), (52) and using (53) we get (50). The lemma is proved.

Equation (50) is all what we need for the rest.

Lemma 4.2. *Let $\tau(z,\nu)$ be a sequence of non-trivial quasiperiodic holomorphic functions on $\mathbb{C}^d$. Suppose that the group $\{2W\nu |\, \nu \in \mathbb{Z}\}$ is Zariski dense in X and equation* (50) *holds. Then there exist wave solutions $\psi(z,\nu,k) = k^\nu \phi(z,\nu,k)$ of the equation* (11) *with u as in* (12) *such that:*

(i) *the coefficients $\xi_s(z,\nu)$ of the formal series*

$$\phi(z,\nu,k) = \xi_0(\nu) + \sum_{s=1}^{\infty} \xi_s(z,\nu)\, k^{-s}$$

are meromorphic functions of the variable $z \in \mathbb{C}^d$ with simple poles at the divisor $\mathcal{T}^\nu$, i.e.,

$$\xi_s(z,\nu) = \frac{\tau_s(z,\nu)}{\tau(z,\nu)}\,, \tag{54}$$

where $\tau_s(z,\nu)$ is now a holomorphic function;

(ii) *$\xi_s(z,\nu)$ satisfy the following monodromy properties*

$$\xi_s(z+\lambda,\nu) - \xi_s(z,\nu) = \sum_{i=1}^{s} B^{\lambda}_{i,\,\nu-s+i}\, \xi_{s-i}(z,\nu)\,, \quad \lambda \in \Lambda\,, \tag{55}$$

where $B^{\lambda}_{i,\,\nu}$ are z-independent.

Proof. The functions $\xi_s(z,\nu)$ are defined recursively by the equations

$$\xi_{s+1}(z-W,\nu) - \xi_{s+1}(z+W,\nu) = u(z,\nu)\,\xi_s(z,\nu-1). \tag{56}$$

The first equation for $s=-1$ is satisfied by an arbitrary z-independent function $\xi_0 = \xi_0(\nu)$. In what follows it will be assumed that $\xi_0(\nu) \neq 0$.

We will now prove lemma by induction in s. Let us assume inductively that for $r \leq s$ the functions ξ_r are known and satisfy (55). Note, that the evaluation of (56) for $s-1$ and $\nu-1$ at the divisor $\mathcal{T}^\nu$ gives the equation

$$\tau_s(z-W)\tau(z+W) = \tau_s(z+W)\tau(z-W)\,, \quad z \in \mathcal{T}^\nu. \tag{57}$$

From (50) and (57) it follows that the two formulae by which we define the residue of ξ_{s+1} on $\mathcal{T}^\nu$

$$\tau^0_{s+1}(z,\nu) = \frac{\tau(z+W,\nu+1)\,\tau_s(z+W,\nu-1)}{\tau(z+2W,\nu)}\,, \quad z \in \mathcal{T}^\nu, \tag{58}$$

$$-\tau^0_{s+1}(z,\nu) = \frac{\tau(z-W,\nu+1)\,\tau_s(z-W,\nu-1)}{\tau(z-2W,\nu)}\,, \quad z \in \mathcal{T}^\nu\,, \tag{59}$$

do coincide.

The expression (58) is certainly holomorphic when $\tau(z+2W)$ is non-zero, i.e., is holomorphic outside of $\mathcal{T}^\nu \cap (\mathcal{T}^\nu - 2W)$. Similarly from (59) we see that $\tau^0_{s+1}(z,\nu)$ is holomorphic away from $\mathcal{T}^\nu \cap (\mathcal{T}^\nu + 2W)$.

We claim that $\tau^0_{s+1}(z,\nu)$ is holomorphic everywhere on $\mathcal{T}^\nu$. Indeed, by assumption the closure of the abelian subgroup generated by $2W$ is everywhere dense. Thus for any $z \in \mathcal{T}^\nu$ there must exist some $N \in \mathbb{N}$ such that $z - 2(N+1)W \notin \mathcal{T}^\nu$; let N moreover be the minimal such N. From (59) it then follows that $\tau^0_{s+1}(z,\nu)$

can be extended holomorphically to the point $z - 2NW$. Thus expression (58) must also be holomorphic at $z - 2NW$; since its denominator there vanishes, it means that the numerator must also vanish. But this expression is equal to the numerator of (59) at $z - 2(N-1)W$; thus τ^0_{s+1} defined from (59) is also holomorphic at $z - 2(N-1)W$ (the numerator vanishes, and the vanishing order of the denominator is one, since we are talking exactly about points on its vanishing divisor). Note that we did not quite need the fact $z - 2(N+1)W \notin \mathcal{T}^\nu$ itself, but the consequences of the minimality of N, i.e., $z - 2kW \in \mathcal{T}^\nu$, $0 \le k \le N$, and the holomorphicity of $\tau^0_{s+1}(z,\nu)$ at $z - 2NW$. Therefore, in the same way, by replacing N by $N-1$, we can then deduce holomorphicity $\tau^0_{s+1}(z,\nu)$ at $z - 2(N-2)W$ and, repeating the process N times, at z.

Recall that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension to all of $\mathbb{C}^d$ ([28]). Therefore, there exists a holomorphic function $\widetilde{\tau}_{s+1}(z,\nu)$ extending the $\tau^0_{s+1}(z,\nu)$. Consider then the function $\chi_{s+1}(z,\nu) = \widetilde{\tau}_{s+1}(z,\nu)/\tau(z,\nu)$, holomorphic outside of $\mathcal{T}^\nu$.

From (55) and (58) it follows that the function

$$f^\lambda_{s+1}(z,\nu) = \chi_{s+1}(z+\lambda,\nu) - \chi_{s+1}(z,\nu) - \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu)$$

has no pole at the divisor $\mathcal{T}^\nu$. Hence it is a holomorphic function. It satisfies the twisted homomorphism relations

$$f^{\lambda+\mu}_{s+1}(z,\nu) = f^\lambda_{s+1}(z+\mu,\nu) + f^\mu_{s+1}(z,\nu)\,,$$

i.e., it defines an element of the first cohomology group of Λ with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda, H^0(\mathbb{C}^d,\mathcal{O}))$. Once again using the same arguments, as that used in the proof of the part (b) of the Lemma 12 in [25], we get that there exists a holomorphic function $h_{s+1}(z,\nu)$ such that

$$f^\lambda_{s+1}(z,\nu) = h_{s+1}(z+\lambda,\nu) - h_{s+1}(z,\nu) + \widetilde{B}^\lambda_{s+1,\,\nu}\xi_0(\nu)\,,$$

where $\widetilde{B}^{\lambda,}_{s+1,\,\nu}$ is z-independent. Hence the function $\zeta_{s+1} = \chi_{s+1} + h_{s+1}$ has the following monodromy properties

$$\zeta_{s+1}(z+\lambda,\nu) - \zeta_{s+1}(z,\nu) = \widetilde{B}^\lambda_{s+1,\nu}\,\xi_0(\nu) + \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu)\,. \quad (60)$$

Let us consider the function R_{s+1} defined by the equation

$$R_{s+1} = \zeta_{s+1}(z-W,\nu) - \zeta_{s+1}(z+W,\nu) - u(z,\nu)\,\xi_s(z,\nu-1)\,. \quad (61)$$

Equation (58) and (59) imply that the right-hand side of (61) has no pole at $\mathcal{T}^\nu \pm W$. Hence $R_{s+1}(z,\nu)$ is a holomorphic function of z. From (55), (60) it follows that it is periodic with respect to the lattice Λ, i.e., it is a function on X. Therefore, R_{s+1} is a constant.

Hence the function

$$\xi_{s+1}(z,\nu) = \zeta_{s+1}(z,\nu) + l_{s+1}(z,\nu)\xi_0(\nu) + c_{s+1}(\nu)\xi_0(\nu)\,,$$

where $c_{s+1}(\nu)$ is a constant, and l_{s+1} is a linear form such that

$$l_{s+1}(2W,\nu)\xi_0(\nu) = -R_{s+1}(\nu)\,,$$

is a solution of (56). It satisfies the monodromy relations (55) with

$$B^{\lambda}_{s+1,\,\nu} = \widetilde{B}^{\lambda}_{s+1,\,\nu} + l_{s+1}(\lambda,\nu)\,.$$

The induction step is completed and thus the lemma is proven.

On each step the ambiguity in the construction of ξ_{s+1} is a choice of linear form $l_{s+1}(z,\nu)$ and constants $c_{s+1}(\nu)$. As in Section 2, we would like to fix this ambiguity by normalizing monodromy coefficients $B^{\lambda}_{i,\,\nu}$ for a set of linear independent vectors $\lambda_1,\dots,\lambda_d \in \Lambda$. As it was revealed in [21] in the fully discrete case there is an obstruction for that. This obstruction is a possibility of the existence of periodic solutions of (56), $\xi_{s+1}(z+\lambda,\nu) = \xi_{s+1}(z,\nu),\ \lambda \in \Lambda$, for $0 \le s \le r-1$.

Note, that there are no periodic solutions of (56) for all s. Indeed, the functions $\xi_s(z,\nu)$ as solutions of non-homogeneous equations are linear independent. Suppose not. Take a smallest nontrivial linear relation among $\xi_s(z,\nu)$, and apply (5.24) to obtain a smaller linear relation. The space of meromorphic functions on X with simple pole at $\mathcal{T}^{\nu}$ is finite-dimensional. Hence there exists minimal r such that equation (56) for $s = r$ has no periodic solutions.

Let $\lambda_1,\dots,\lambda_d$ be a set of linear independent vectors in Λ. Without loss of generality throughout the rest of the paper it will be assumed that there is no linear form $l(z),\ z \in \mathbb{C}^d$, with $l(\lambda_j) = 1$ and $l(2W) = 0$.

Lemma 4.3. *Suppose equations* (56) *has periodic solutions for $s < r$ and has a quasi-periodic solution ξ_r whose monodromy relations for λ_j have the form*

$$\xi_r(z+\lambda_j,\nu) - \xi_r(z,\nu) = b\,\xi_0(\nu)\,,\quad j = 1,\dots,d\,,$$

where $b \neq 0$ is a constant. Then for all s equations (56) *has solutions of the form* (54) *satisfying* (55) *with $B^{\lambda_j}_{i,\,\nu} = b\,\delta_{i,r}$, i.e.,*

$$\xi_s(z+\lambda_j,\nu) - \xi_s(z,\nu) = b\,\xi_{s-r}(z,\nu)\,.$$

Proof. We will now prove the lemma by induction in $s \ge r$. Let us assume inductively that ξ_{s-r} is known, and for $1 \le i \le r$ there are solutions $\tilde{\xi}_{s-r+i}$ of (56) satisfying (55) with $B^{\lambda_j}_{i,\,\nu} = b\,\delta_{i,r}$. Then, according to the previous lemma, there exists a solution $\tilde{\xi}_{s+1}$ of (56) having the form (54) and satisfying monodromy relations (55), which for λ_j have the form

$$\tilde{\xi}_{s+1}(z+\lambda_j,\nu) - \tilde{\xi}_{s+1}(z,\nu) = b\,\tilde{\xi}_{s-r+1}(z,\nu) + B^{\lambda_j}_{s+1,\,\nu}\xi_0(\nu)\,.$$

If ξ_{s-r} is fixed, then the general quasi-periodic solution ξ_{s-r+1} with the normalized monodromy relations is of the form

$$\xi_{s-r+1}(z,\nu) = \widetilde{\xi}_{s-r+1}(z,\nu) + c_{s-r+1}(\nu)\xi_0(\nu)\,. \tag{62}$$

It is easy to see that under the transformation (62) the functions $\widetilde{\xi}_{s-r+i}$ get transformed to

$$\xi_{s-r+i}(z,\nu) = \widetilde{\xi}_{s-r+i}(z,\nu) + c_{s-r+1}(\nu-i+1)\,\xi_{i-1}(z,\nu)\,.$$

This transformation does not change the monodromy properties of ξ_{s-r+i} for $i \le r$, but changes the monodromy property of ξ_{s+1}:

$$\xi_{s+1}(z+\lambda_j,\nu) - \xi_{s+1}(z,\nu) \\ = b\,\xi_{s-r+1}(z,\nu) + B^{\lambda_j}_{s+1,\,\nu}\,\xi_0(\nu) + b\,(c_{s-r+1}(\nu-r) - c_{s-r+1}(\nu))\,\xi_0(\nu)\,.$$

Recall, that $\widetilde{\xi}_{s+1}$ was defined up to a linear form $l_{s+1}(z,\nu)$ which vanishes on $2W$. Therefore the normalization of the monodromy relations for ξ_{s+1} uniquely defines this form and the differences $(c_{s-r+1}(\nu-r) - c_{s-r+1}(\nu))$. The induction step is completed and the lemma is thus proven.

Note, the following important fact: if ξ_{s-r} is fixed then ξ_{s-r+1}, such that there exists quasi-periodic solution ξ_{s+1} with normalized monodromy properties, is defined uniquely up to the transformation:

$$\xi_{s-r+1}(z,\nu) \longmapsto \xi_{s-r+1}(z,\nu) + c_{s-r+1}(\nu)\xi_0(\nu), \quad c_{s-r+1}(\nu+r) = c_{s-r+1}(\nu)\,. \tag{63}$$

Our next goal is to show that the assumption of Lemma 4.3 holds for some r, and then to fix the remaining ambiguity (63) in the definition of the wave function. At this moment we are going to use for the first time the assumption that τ is a meromorphic periodic function of the variable ν.

Let r be the minimal integer such that there exist solutions $\xi^0_0 = 1$, ξ^0_1, …, ξ^0_{r-1} of (56) that are periodic functions of z with respect to Λ, and there is no periodic solution ξ_r of (56). As it was noted above, the functions τ_s are linear independent. Hence $r \le h^0(Y, \theta|_Y)$.

If ξ^0_{r-1} is periodic, then the monodromy relation for ξ_r has the form

$$\xi^0_r(z+\lambda,\nu) - \xi^0_r(z,\nu) = B^\lambda_r(z,\nu)\,, \quad \lambda \in \Lambda\,. \tag{64}$$

The function B^λ_r is independent of the ambiguities in the definition of ξ_i, $i < r$, and therefore, it is a well-defined holomorphic function of $z \in X$. Hence it is z-independent, $B^\lambda_r(z,\nu) = B^\lambda_r(\nu)$. The function ξ^0_r is defined up to addition of a linear form $l_r(z,\nu)$ such that $l(2W,\nu) = 0$. Therefore, there exist the solution ξ^0_r such that $B^{\lambda_j}_r(\nu) = B_r(\nu)$. There is no ξ^0_r which is periodic for all ν. Hence $B_r(\nu) \ne 0$ at least for one value of ν. By assumption the function τ is a meromorphic function of ν. Therefore, $B_r(\nu)$ is a meromorphic function of ν. Shifting $\nu \to \nu + \nu_0$ if needed, we may assume without loss of generality that $B_r(\nu) \ne 0$ for *all* $\nu \in \mathbb{Z}$. From (15) it follows that $u(z,\nu+N) = u(z,\nu)$. Hence $B_r(\nu)$ is a periodic function of ν, i.e.,

$$B_r(\nu+N) = B_r(\nu)\,.$$

Under the transformation

$$\xi^0_0 = 1 \longmapsto \xi_0(\nu)$$

the solutions ξ^0_r get transformed to

$$\xi_s(z,\nu) = \xi^0_s(z,\nu)\,\xi_0(\nu-s)\,.$$

From (64) it follows that the transformed function ξ_r satisfies the relations

$$\xi_r(z+\lambda,\nu) - \xi_r(z,\nu) = B^\lambda_r(\nu)\xi_0(z,\nu-r)\,, \quad \lambda \in \Lambda\,.$$

The equation

$$b\,\xi_0(\nu) = B_r(\nu)\xi_0(\nu - r), \quad \xi_0(\nu + N) = \xi_0(\nu)\,. \tag{65}$$

restricted to the space of periodic functions ξ_0 can be regarded as a finite-dimensional linear equation. The vanishing of the determinant of this equation defines the constant b. With b fixed equation (65) defines ξ_0 uniquely up to multiplication by a function $c_0(\nu)$ such that $c_0(\nu+N) = c_0(\nu+r) = c_0(\nu)$. By the assumption of Theorem 1.2 the period N is prime and $N > H^0(\mathcal{T}^\nu)$. As it was mentioned above $r \le H^0(\mathcal{T}^\nu)$. Hence two periods of c_0 are coprime, i.e., $(r, N) = 1$. Therefore, ξ_0 is defined uniquely up to a constant factor.

Lemma 4.4. *Suppose that the assumptions of Theorem* 1.2 *hold. Then there exists a formal solution*

$$\phi = \xi_0(\nu) + \sum_{s=1}^{\infty} \xi_s(z,\nu)\, k^{-s}$$

of the equation

$$k\phi(z - W, \nu, k) = k\phi(z + W, \nu, k) + u(z,\nu)\,\phi(z, \nu - 1, k)\,, \tag{66}$$

with u as in (12) *such that:*

(i) *the coefficients ξ_s of the formal series ϕ are of the form $\xi_s = \tau_s/\theta$, where $\tau_s(z)$ are holomorphic functions;*

(ii) *$\phi(z,\nu,k)$ is quasi-periodic with respect to the lattice Λ and for the basis vectors λ_j in $\mathbb{C}^d$ its monodromy relations have the form*

$$\phi(z + \lambda_j, \nu, k) = (1 + b\,k^{-r})\,\phi(z,\nu,k), \quad j = 1, \dots, d\,,$$

where b are constants defined by (65)*;*

(iii) *$\phi(z,\nu,k)$ is a quasi-periodic function of the variable ν, i.e.,*

$$\phi(z, \nu + N, k) = \phi(z,\nu,k)\mu(k)\,; \tag{67}$$

(iv) *ϕ is unique up to the multiplication by a constant in z factor $\rho(k)$.*

Proof. We prove the lemma by induction in s. Let us assume inductively that ξ_{s-r} is known. As shown above the normalization of the relations for ξ_{s+1} uniquely defines ξ_{s-r+1} up to the transformation (63), i.e., up to a r-periodic function $c_{s-r+1}(\nu+r) = c_{s-r+1}(\nu)$. The quasiperiodicity condition (iii) is equivalent to the condition that this function of c_{s-r+1} is N-periodic. As it was mentioned above the periods r and N are coprime. Hence on each step ξ_{s-r+1} is defined up to the additive constant. This ambiguity corresponds to the multiplication of ϕ be a constant factor $\rho(k)$, and thus the lemma is proven.

4.2. Commuting difference operators

As in Section 3 we are now going to construct rings $\mathcal{A}^z$ of commuting difference operators. First we introduce pseudo-difference operator in one of the original variables m depending on the second variable n and a point $z \in \mathbb{C}^d$. Recall that

the variables n, m are related to x, ν via (10). In what follows we will denote

$$T = e^{\partial_m}, \quad T_1 = e^{\partial_n}, \quad \Delta = T - 1\,, \quad \Delta_1 = T_1 - 1\,, \quad \Delta_0 = T_1 - T\,.$$

The formal series $\phi(z, \nu, k)$ defines a unique pseudo-difference operator

$$\mathcal{L}(z, \nu) = w_0(\nu)T + \sum_{s=0}^{\infty} w_{s+1}(z, \nu)\, T^{-s},$$

such that the equation

$$\left(w_0(m+n)T + \sum_{s=0}^{\infty} w_s(z + (m-n)W, (m+n))\, T^{-s}\right)\psi = k\psi \tag{68}$$

holds. Here $\psi = k^{n+m}\phi(z + (m-n)W, (m+n), k)$. The coefficients $w_s(z, \nu)$ of $\mathcal{L}$ are difference polynomials in terms of the coefficients of ϕ. Due to quasiperiodicity of ψ they are meromorphic functions on the abelian variety X.

From equations (66), (68) it follows that

$$\left((\Delta_1\mathcal{L}^i)\, T_1 - (\Delta\mathcal{L}^i)\, T - [u, \mathcal{L}^i]\right)\psi = 0\,,$$

where $\Delta_1\mathcal{L}^i$ and $\Delta\mathcal{L}^i$ are pseudo-difference operator in T, whose coefficients are difference derivatives of the coefficients of $\mathcal{L}^i$ in the variables n and m respectively. Using the equation $(T_1 - T - u)\,\psi = 0$, we get

$$\left(\left(\Delta_1\mathcal{L}^i\right) T - \left(\Delta\mathcal{L}^i\right) T + \left(\Delta_1\mathcal{L}^i\right) u - [u, \mathcal{L}^i]\right)\psi = 0\,. \tag{69}$$

The operator in the left-hand side of (69) is a pseudo-difference operator in the variable m. Therefore, it has to be equal to zero. Hence we have the equation

$$\left(\Delta_0\mathcal{L}^i\right) T + \left(\Delta_1\mathcal{L}^i\right) u - [u, \mathcal{L}^i] = 0\,.$$

Let $\mathcal{L}^i_+$ be the strictly positive difference part of the operator $\mathcal{L}^i$, i.e.,

$$\mathcal{L}^i = \mathcal{L}^i_+ + \mathcal{L}^i_- = \mathcal{L}^i_+ + \sum_{s=0}^{\infty} F_{i,s}T^{-s}\,. \tag{70}$$

Then

$$\left(\Delta_0\mathcal{L}^i_+\right) T + \left(\Delta_1\mathcal{L}^i_+\right) u - [u, \mathcal{L}^i_+] = -\left(\Delta_0\mathcal{L}^i_-\right) T - \left(\Delta_1\mathcal{L}^i_-\right) u + [u, \mathcal{L}^i_-]\,. \tag{71}$$

The left-hand side of (71) is a difference operator with non-vanishing coefficients only at the positive powers of T. The right-hand side is a pseudo-difference operator of order 1. Therefore, it has the form f_iT. The coefficient f_i is easily expressed in terms of the leading coefficient $\mathcal{L}^i_-$. Finally we get the equation

$$\left(\Delta_0\mathcal{L}^i_+\right) T + \left(\Delta_1\mathcal{L}^i_+\right) u - [u, \mathcal{L}^i_+] = -(\Delta_0 F_i)\, T\,, \tag{72}$$

where $F_i = F_i = \operatorname{res}\ \mathcal{L}^i$.

By definition of $\mathcal{L}$ we have that the functions F_i in (70) are of the form

$$F_i = \operatorname{res}_T \mathcal{L}^i = F_i(z + (m-n)W, (m+n))\,,$$

where for each ν the functions $F_i(z, \nu)$ are abelian functions, i.e., periodic functions of the variable $z \in \mathbb{C}^d$.

Lemma 4.5. *The abelian functions F_i have the form*

$$F_i(z,\nu) = \frac{q_i(z+W,\nu+1)}{\tau(z+W,\nu+1)} - \frac{q_i(z,\nu)}{\tau(z,\nu)}\,, \tag{73}$$

where $q_i(z,\nu)$ are holomorphic functions of the variable $z \in \mathbb{C}^d$.

Proof. The wave solution ψ defines a unique operator Φ such that

$$\psi = \Phi k^{n+m}, \quad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s\big((m-n)W + z, m+n\big)\, T^{-s}\,,$$

where $\varphi_s(z,\nu)$ are meromorphic functions of $z \in \mathbb{C}^d$. The dual wave function

$$\psi^+ = k^{-n-m}\left(1 + \sum_{s=1}^{\infty} \xi_s^+\big((n-m)W + z, n+m\big)\, k^{-s}\right)$$

is defined by the formula

$$\psi^+ = k^{-n-m}\, T_1\, \Phi^{-1}\, T_1^{-1}.$$

It satisfies the equation

$$(T_1^{-1} - T^{-1} - u)\, \psi^+ = 0\,,$$

which implies that the functions ξ_s^+ have the form $\xi_s^+(z,\nu) = \tau_s^+(z,\nu)/\theta(z+W, \nu+1)$, where $\tau_s^+(z,\nu)$ are holomorphic functions of $z \in \mathbb{C}^d$. Therefore, the functions $J_s(z,\nu)$ such that

$$(\psi^+ T_1)\, \psi = k + \sum_{s=1}^{\infty} J_s((n-m)W + z, (n+m))\, k^{-s+1}$$

are meromorphic function on X with the simple poles at $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu+1} - W$.

The same arguments as that used for the proof of (44) show that

$$(\psi^+ T_1)\psi = (k^{-x} T_1 \Phi^{-1})(\Phi k^x) = k + (\Delta Q)\,,$$

where the coefficients of the series Q are of the form

$$Q = \sum_{s=0}^{\infty} Q_s\big((n-m)W + z, n+m\big) k^{-s}\,,$$

and the functions $Q_s(z,\nu)$ are difference polynomials in the coefficients φ_s of the wave operator. Therefore, they are well-defined meromorphic functions of z. As shown above, the functions

$$J_s(z,\nu) = Q_s(z+W,\nu+1) - Q_s(z,\nu) \tag{74}$$

have simple poles at $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu+1} - W$. Hence $Q_s(z,\nu)$ has poles only at $\mathcal{T}^\nu$, i.e.,

$$Q_s = \frac{q_s(z,\nu)}{\tau(z,\nu)}\,, \tag{75}$$

where $q_s(z,\nu)$ are holomorphic functions of z.

From the definition of $\mathcal{L}$ it follows that

$$\operatorname{res}_k \left((\psi^+ T_1) \, (\mathcal{L}^i \psi) \right) k^{-2} dk = \operatorname{res}_k \left((\psi^+ \, T_1) \, \psi \right) k^{i-2} dk = J_i \, . \tag{76}$$

On the other hand, using (45) we get

$$\operatorname{res}_k (\psi^+ \, T_1) \, (\mathcal{L}^i \psi) \, k^{-2} dk = \operatorname{res}_k \left(k^{-n-m} \Phi^{-1} \right) \left(\mathcal{L}^i \Phi k^{n+m} \right) k^{-1} dk = \operatorname{res}_T \mathcal{L}^i = F_i \, . \tag{77}$$

Equation (73) is a direct corollary of (74)–(77). The lemma is proved.

The function ψ is quasiperiodic function of the variable ν. Then, from the definition of ψ^+ it follows that

$$\phi^+(z, \nu + N, k) = \phi^+(z, \nu, k) \mu^{-1}(k) \, ,$$

where $\mu(k)$ is defined in (67). Therefore, the functions J_s are periodic functions of ν. Hence

$$F_i(z, \nu + N) = F_i(z, \nu) \, .$$

For each ν the space of functions spanned by the abelian functions $F_i(z, \nu)$ is finite-dimensional. Due to periodicity of F_i in ν the total space $\mathcal{F}$ spanned by sequences $F_i(z, \nu)$ is also finite-dimensional. Let $\{F_\alpha \mid \alpha \in A\}$, for finite set A, be a basis of the factor- space of $\mathcal{F}$ modulo z-independent sequences. Then for all $i \notin A$ there exist constants $c_{i,\alpha}, d_i(\nu)$ such that

$$F_i(z, \nu) - \sum_{\alpha \in A} c_{i,\alpha} F_\alpha(z, \nu) = d_i(\nu) \, . \tag{78}$$

The rest of the proof of Theorem 1.2 is identical to that in the proof of Theorem 1.1. Namely,

Lemma 4.6. *Let ψ be a wave function corresponding to u, and let L_i, $i \notin A$, be the difference operator given by the formula*

$$L_i = \mathcal{L}_+^i - \sum_{\alpha \in A} c_{i,\alpha} \mathcal{L}_+^\alpha \, , \ i \notin A \, ,$$

where the constants $c_{i,\alpha}$ are defined by equations (78).

Then the equations

$$L_i \, \psi = a_i(k) \, \psi, \quad a_i(k) = k^i + \sum_{s=1}^{\infty} a_{s,i} k^{n-s} \, ,$$

where $a_{s,i}$ are constants, hold.

Proof. From (72) it follows that

$$[T_1 - T - u, L_i] = 0 \, .$$

Hence, if ψ is the wave solution of (11), then $L_i \psi$ is also a wave solution of the same equation. By uniqueness of the wave function up to a constant in z-factor we get (48) and thus the lemma is proven.

Corollary 4.1. *The operators L_i^z commute with each other,*

$$[L_i^z, L_j^z] = 0 \, .$$

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A Special Case of the Γ_{00} Conjecture

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Abstract. In this paper we prove the Γ_{00} conjecture of van Geemen and van der Geer [8] under the additional assumption that the matrix of coefficients of the tangent has rank at most 2 (see Theorem 1 for a precise formulation). This assumption is satisfied by Jacobians (see proposition 1), and thus our result gives a characterization of the locus of Jacobians among all principally polarized abelian varieties.

The proof is by reduction to the (stronger version of the) characterization of Jacobians by semidegenerate trisecants, i.e., by the existence of lines tangent to the Kummer variety at one point and intersecting it in another, proven by Krichever in [16] in his proof of Welters' [20] trisecant conjecture.

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The Schottky problem is the question of characterizing Jacobians of algebraic curves among all ppavs – complex principally polarized abelian varieties (A, Θ). Many approaches and solutions (or weak solutions – those that only characterize the Jacobian locus up to other irreducible components) to the problem have been developed, via the singular locus of the theta divisor (Andreotti-Mayer [1]), representability of the curves of the minimal class (Matsusaka [17] and Ran [18]), the Kadomtsev-Petviashvili equation (Shiota [19]), etc.

One approach that led to various characterizations of Jacobians, culminating in Krichever's proof [16] of Welters' trisecant conjecture [20], is via the geometry of the linear system $|2\Theta|$. The linear subsystem $\Gamma_{00} \subset |2\Theta|$ is defined to consist of those sections that vanish with multiplicity at least 4 at the origin.

It was essentially known to Frobenius, and more recently proven by van Geemen and van der Geer [8], Dubrovin [5], Fay [7], Gunning [11] that for Jacobians of Riemann surfaces all elements of Γ_{00} vanish along $C - C$. This led van Geemen and van der Geer to make the following

Γ_{00} Conjecture. ([8]) *If for an indecomposable (not a product of lower-dimensional) ppav (A, Θ) the base locus $Bs(\Gamma_{00}) \neq \{0\}$, then (A, Θ) is a Jacobian.*

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Conjecture 2 in [8] is that A is a Jacobian if $\dim Bs(\Gamma_{00}) \geq 2$, but the stronger version stated above was further conjectured – see the discussion by Donagi in [4]. In [8], conjecture 1, van Geemen and van der Geer further conjectured that for a Jacobian of a curve C the base locus $Bs(\Gamma_{00})$ is in fact equal to $C - C$ (only containment was known classically). This equality on the level of sets was proven for $g > 4$ by Welters in [21] (he also showed that for $g = 4$ the base locus of Γ_{00} contains two more points). Izadi in [12] showed that $Bs(\Gamma_{00}) = C - C$ holds scheme-theoretically.

Beyond providing a solution to the Schottky problem, the validity of the Γ_{00} conjecture would have implications for example for characterization of hyperelliptic Jacobians by their Seshadri constants (see Debarre [3]). The Γ_{00} conjecture was proven for a generic ppav for $g = 5$ or $g \geq 14$ by Beauville, Debarre, Donagi, and van der Geer in [2]. Izadi proved in [13] the conjecture for all 4-dimensional ppavs, and obtained further results on $Bs(\Gamma_{00})$ for Prym varieties in [14].

In fact a slightly more precise result is known for Jacobians: from the explicit formulas for Jacobians in [7] and [11] more information can be obtained than just the statement that $C - C \subset Bs(\Gamma_{00})$ (see Proposition 1 for the precise formulation). In this paper we prove that this refined version of the Γ_{00} conjecture (a certain extra condition on the linear dependence, see Theorem 1 for a precise formulation) characterizes Jacobians. The proof is by reducing this conjecture to a characterization of Jacobians by the validity of a certain difference-differential equation for the theta function on the theta divisor, proven by Krichever in [16] along the way of his proof of the semidegenerate case of the trisecant conjecture.

1. Γ_{00} conjecture as a condition on the Kummer variety

We now review the terminology and notations regarding the $|2\Theta|$ linear system and the Kummer variety, and formulate the precise version of the Γ_{00} conjecture that we prove. There are no new results in this section; we gather the notations, and give proofs for completeness.

Given a ppav (A, Θ) the Kummer map is the embedding

$$|2\Theta| : A/\pm 1 \hookrightarrow \mathbb{P}^{2^g-1}$$

and the Kummer variety is the image of this map. We denote the Kummer variety $K(A)$ and the Kummer embedding map by K. It is often convenient to choose a specific basis of the space of sections $H^0(A, 2\Theta)$. If the abelian variety is given as $A = \mathbb{C}^g/\mathbb{Z}^g + B\mathbb{Z}^g$ for some symmetric $g \times g$ matrix B with positive-definite imaginary part, and the polarization Θ is the divisor of the theta function

$$\theta(B, z) := \sum_{n \in \mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bn,n)},$$

where $(\cdot,\cdot)$ denotes the complex scalar product (not Hermitian!) of vectors in $\mathbb{C}^g$, then the basis of $H^0(A, 2\Theta)$ consists of

$$\Theta[\epsilon](B,z) := \sum_{n\in\mathbb{Z}^g} e^{2\pi i(2n+\epsilon,z)+\pi i\left(2n+\epsilon,B(n+\frac{\epsilon}{2})\right)}$$

for all $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^g$. We will suppress B in the notations, as it will not vary in our discussion.

Fay [6] and Gunning [10, 11] observed that the Kummer image of a Jacobian has many trisecant lines. Gunning [10] proved that the existence of a certain one-dimensional family of trisecants characterizes Jacobians; Welters [20] proved that a jet of such a curve suffices, and conjectured that the existence of a single trisecant already characterizes Jacobians. In view of this conjecture one can also ask whether degenerate trisecants characterize Jacobians: here the most degenerate trisecant corresponds to the case of a flex line, and the semidegenerate case is that of a line tangent to the Kummer image and intersecting it in some other point. The flex line case of the trisecant conjecture was proven by Krichever in [15], and the remaining two cases – in [16].

Notice that all sections of $|2\Theta|$ are even functions on the universal cover $\mathbb{C}^g$ of the abelian variety. Thus a section of $|2\Theta|$ vanishes at zero to order 4 if and only if it vanishes together with all its second derivatives, as the first and third order derivatives are automatically zero. We thus have

$$\Gamma_{00} = \{f \in |2\Theta| \ : \ 0 = f(0) = \partial_{z_i}\partial_{z_j} f(0) = 0 \ \forall i,j\},$$

where we denote differentiation by $\partial_z = \frac{\partial}{\partial z}$. Since any section $f \in |2\Theta|$ is a linear combination $f = \sum f_\epsilon \Theta[\epsilon]$, the conditions for f to lie in Γ_{00} are linear conditions on the coefficients f_ϵ, and we get

$$Bs(\Gamma_{00}) = \left\{ z \in A \ : \ K(z) \in \left\langle K(0), \partial_{z_i}\partial_{z_j} K(0) \right\rangle_{\text{linear span}} \right\}.$$

Thus we get a reformulation (given in [8])

Γ_{00} Conjecture (equivalent reformulation). *If for an indecomposable ppav (A,Θ) there exists a point $P \in A \setminus \{0\}$, and some numbers $c, c_{ij} \in \mathbb{C}$, for $1 \le i,j \le g$ such that*

$$K(P) = cK(0) + \sum_{i,j=1}^{g} c_{ij}\partial_{z_i}\partial_{z_j} K(0),$$

then (A,Θ) is a Jacobian.

Notice that since for taking partial derivatives the order of operations does not matter, we can assume that the matrix c_{ij} is symmetric. Moreover, is is known that for Jacobians the matrix c_{ij} is in fact of rank 1 (or if we want it to be symmetric, of rank 2). In our notations the result is the following

Proposition 1 (Dubrovin [5], Fay [7] theorem 2.5, van Geemen and van der Geer [8] proposition 2.1, Gunning [11] corollary 1, essentially known to Frobenius)**.** *Choose any points p, q on the Abel-Jacobi image of a curve C in its Jacobian* $\mathrm{Jac}(C)$*, and*

let U, V be the tangent vectors to the image of C at p and q respectively. Then there exist constants $b, c \in \mathbb{C}$ such that

$$K(p-q) = cK(0) + b\partial_U\partial_V K(0).$$

Proof. We sketch the proof of this proposition to emphasize the relation (but a priori not an equivalence) of this statement and the existence of trisecants. To prove this, one uses a suitable degeneration of Fay's trisecant formula [6, 10]. Indeed, it is known that for arbitrary points p, p_1, p_2, p_3 on the Abel-Jacobi image of a curve the 3 points

$$K\left(\frac{p+p_1-p_2-p_3}{2}\right), K\left(\frac{p+p_2-p_3-p_1}{2}\right), K\left(\frac{p+p_3-p_1-p_2}{2}\right)$$

are collinear, i.e., linearly dependent. The "semidegenerate" case of this conjecture is when such a trisecant line degenerates to a tangent line to the Abel-Jacobi image at some point, intersecting it also in some other point, i.e., is the case when, say, $p_2, p_3 \to q$ for p, p_1 fixed. In this case the limiting statement is that

$$K\left(\frac{p+p_1-2q}{2}\right), K(p-p_1), \partial_U K(p-p_1)$$

are linearly dependent, where U is the tangent vector at q to $C \subset \mathrm{Jac}(C)$. If we further degenerate this condition as $p \to p_1$, and again take the first term in the Taylor expansion of $\partial_U K(p-p_1)$ (which, notice, vanishes at 0 by parity), in the limit we get the statement that

$$K(p-q), K(0), \partial_V\partial_U K(0),$$

are collinear, where V is the tangent vector at p to $C \subset \mathrm{Jac}(C)$, which is exactly the statement of the proposition. □

Remark. Note that while we can degenerate a family of trisecants, there is no a priori reason why the degenerate condition would imply the less degenerate one, and thus it is not at all clear that the statement of the Γ_{00} conjecture implies the existence of (semidegenerate) trisecants. Similarly, there is no a priori way to show that the existence of just one less degenerate trisecant implies the existence of any more degenerate ones.

The purpose of this article is to prove that this version of the Γ_{00} conjecture – if we additionally require the symmetric matrix $\{c_{ij}\}$ to have rank at most two – characterizes Jacobians. The fact that a similar condition (with $U = V$) is related to the KP equation is discussed by van Geemen and van der Geer [8], observation 4.10.

Theorem 1 (Main theorem). *This special case (the symmetric matrix $\{c_{ij}\}$ having rank at most two, so that we can write $\{c_{ij}\} = (U \otimes V + V \otimes U)/2$) of the Γ_{00} conjecture characterizes Jacobians, i.e., an indecomposable ppav (A, Θ) is a Jacobian if and only if there exist a point $P \in A \setminus \{0\}$, vectors $U, V \in \mathbb{C}^g$, and a constant $c \in \mathbb{C}$ such that*

$$K(P) = cK(0) + \partial_U\partial_V K(0). \tag{1}$$

2. Γ_{00} conjecture as a difference-differential equation on the theta divisor

Similar to the ideas of [15], [16], and [9] we first use Riemann's bilinear addition theorem

$$\theta(z+Z)\theta(z-Z) = \sum_{\epsilon\in(\mathbb{Z}/2\mathbb{Z})^g} \Theta[\epsilon](z)\Theta[\epsilon](Z) =: K(z)\cdot K(Z) \tag{2}$$

(where the dot denotes the scalar product in $\mathbb{C}^{2^g}$, and we consider the lifting of the map K to the universal cover $\mathbb{C}^g$ of A) to rewrite equation (1) as a functional equation for the theta function. Indeed, for arbitrary $z\in\mathbb{C}^g$ let us take the scalar product of (1) with $K(z)$ and apply (2) on both sides to get

$$\theta(z+P)\theta(z-P) = c\theta^2(z) + K(z)\cdot\partial_U\partial_V K(0).$$

To express the last term in the equation in terms of the theta function, we take the $\partial_U\partial_V$ derivative of (2) in the Z variable, obtaining

$$K(z)\cdot\partial_U\partial_V K(Z) = \partial_U\partial_V\theta(z+Z)\,\theta(z-Z) + \theta(z+Z)\,\partial_U\partial_V\theta(z-Z)$$
$$-\partial_U\theta(z+Z)\partial_V\theta(z-Z) - \partial_V\theta(z+Z)\partial_U\theta(z-Z).$$

Setting now $Z=0$ we get

$$K(z)\cdot\partial_U\partial_V K(0) = 2\theta(z)\partial_U\partial_V\theta(z) - 2\partial_U\theta(z)\partial_V\theta(z).$$

We now substitute this expression back in to get

$$\theta(z+P)\theta(z-P) = c\theta^2(z) + 2\theta(z)\partial_U\partial_V\theta(z) - 2\partial_U\theta(z)\partial_V\theta(z).$$

We will now replace the argument z by $Ux+Vy+Pt+Z$, for $x,y,t\in\mathbb{C}$, and denote

$$\tau(x,y,t) := \theta(Ux+Vy+Pt+Z); \tag{3}$$
$$u(x,y,t) := 2\partial_x\partial_y \ln\tau(x,y,t), \tag{4}$$

so that the above equation becomes

$$\tau(x,y,t+1)\tau(x,y,t-1) = \tau^2(x,y,t)\,(c+u(x,y,t))\,.$$

Dividing through by $\tau(x,y,t)\tau(x,y,t-1)$ and denoting

$$\psi(x,y,t) := \frac{\tau(x,y,t)}{\tau(x,y,t-1)} \tag{5}$$

we get the following

Proposition 2. *The assumption of the main theorem, i.e., equation* (1) *having a solution, is equivalent to the equation*

$$(c+u(x,y,t)-T)\psi(x,y,t) = 0 \tag{6}$$

being satisfied with $\tau, u,$ *and* ψ *given by* (3), (4), (5), *with* $T = e^{\partial_t}$ *being the operator of shifting* t *by* 1.

Similar to the previous work on the subject, we now investigate the condition for equation (6) to have any solution with $\psi(x,y,t)$ having a simple pole along the divisor of $\tau(x,y,t-1)$ – without requiring ψ to have the exact form of (5).

Proposition 3. *If for an indecomposable ppav equation* (1) *is satisfied then for any* z *on the theta divisor*

$$\theta_{UU}(z)\theta(z-P)\theta(z+P) = \theta_U(z)\theta_U(z-P)\theta(z+P) + \theta_U(z)\theta_U(z+P)\theta(z-P), \quad (7)$$

where the subscript U *denotes the directional derivative in direction* U *with respect to* z.

Proof. To show this, suppose that the function $\tau(x,y,t)$ locally has a root at $x=\eta(y,t)$, where $\eta(y,t)$ is some function, and suppose that locally $\tau(\eta(y,t),y,t-1)\tau(\eta(y,t),y,t+1)\neq 0$ (this is true generically on the theta divisor Θ, since for indecomposable ppavs we would otherwise have $\Theta=\Theta+P$ or $\Theta=\Theta-P$, but the corresponding line bundles are not linearly equivalent). Then locally near the point $x=\eta(y,t)$ from the Taylor expansion

$$\tau(x,y,t) = \alpha(x-\eta) + O((x-\eta)^2)$$

where $\alpha\in\mathbb{C}$ we get from (4) the Taylor series

$$\begin{aligned} u(x,y,t) &= 2\partial_x\partial_y \ln\tau(x,y,t) = 2\partial_x\partial_y \ln(\alpha(x-\eta)+O((x-\eta)^2) \\ &= 2\partial_y\left(\frac{1}{x-\eta}+O(1)\right) = \frac{2\eta_y}{(x-\eta)^2}+O(1), \end{aligned}$$

while from (5) we obtain

$$\begin{aligned} \psi(x,y,t) &= \frac{\tau(x,y,t)}{\tau(x,y,t-1)} = a(x-\eta)+b(x-\eta)^2+O((x-\eta)^3), \\ T\psi(x,y,t) &= \frac{\tau(x,y,t+1)}{\tau(x,y,t)} = \frac{A}{x-\eta}+B+O(x-\eta), \end{aligned}$$

where a,b,A,B are in fact some values of derivatives that we will eventually need to compute. When we substitute all of these expansions into the left-hand side of (6) and expand, we get

$$(c+u-T)\psi = \frac{2\eta_y}{(x-\eta)^2}\,a(x-\eta) - \frac{A}{x-\eta} + 2b\eta_y - B + O(x-\eta).$$

By equation (6) this is identically zero, and so we must have

$$2a\eta_y = A; \qquad 2b\eta_y = B.$$

We eliminate η_y from these expressions to get the relation $Ab=aB$. Now compute the coefficients a,b,A,B in the Taylor expansions. It is notationally convenient to go back from τ to θ at this point, noting that $\tau_x=\theta_U$, and to denote $z:=Ux+Vy+Pt+Z$ the point lying on the theta divisor (for $x=\eta(y,t)$).

We then have by differentiating (5)

$$a = \partial_x\psi(x,y,t)|_{x=\eta(y,t)} = \partial_x\frac{\tau(x,y,t)}{\tau(x,y,t-1)}|_{x=\eta(y,t)}$$

$$= \partial_U \frac{\theta(z)}{\theta(z-P)}|_{\{\theta(z)=0\}} = \frac{\theta_U(z)}{\theta(z-P)},$$

$$b = \partial_x \partial_x \psi(x,y,t)|_{x=\eta(y,t)} = \partial_U \left(\frac{\theta_U(z))}{\theta(z-P)} - \frac{\theta_U(z-P)\theta(z)}{\theta^2(z-P)} \right) |_{\{\theta(z)=0\}}$$

$$= \frac{\theta_{UU}(z)}{\theta(z-P)} - 2\frac{\theta_U(z-P)\theta_U(z)}{\theta^2(z-P)}$$

(notice that one term vanished in each formula since $\tau(\eta, y, t) = 0$). To compute A and B, it is convenient to think of the expansion of

$$(T\psi(x,y,t))^{-1} = \frac{\tau(\eta,y,t)}{\tau(\eta,y,t+1)},$$

which is the same as above with $z-P$ replaced by $z+P$, while on the other hand it is given by

$$(T\psi(x,y,t))^{-1} = \frac{1}{A}(x-\eta) - \frac{B}{A^2}(x-\eta) + O((x-\eta)^3).$$

This yields

$$\frac{1}{A} = \frac{\theta_U(z)}{\theta(z+P)}; \quad -\frac{B}{A^2} = \frac{\theta_{UU}(z)}{\theta(z+P)} - 2\frac{\theta_U(z+P)\theta_U(z)}{\theta^2(z+P)}$$

Substituting all of these back into $Ab = aB$, which it is convenient to write as $\frac{b}{a} = A\frac{B}{A^2}$, yields

$$\frac{\theta(z-P)}{\theta_U(z)} \left(\frac{\theta_{UU}(z)}{\theta(z-P)} - 2\frac{\theta_U(z-P)\theta_U(z)}{\theta^2(z-P)} \right)$$
$$= -\frac{\theta(z+P)}{\theta_U(z)} \left(\frac{\theta_{UU}(z)}{\theta(z+P)} - 2\frac{\theta_U(z+P)\theta_U(z)}{\theta^2(z+P)} \right)$$

which upon clearing the denominators gives the required identity on the theta divisor $\{\theta(z) = 0\}$. □

Proof of the main theorem. We note that up to replacing U by V equation (7) is identical to equation (1.7) in [16], which is shown there to be equivalent to the semidegenerate case of the trisecant conjecture. In [16] it is then shown that the condition of having a semidegenerate trisecant characterizes Jacobians, and thus (7) is equivalent to the ppav being a Jacobian. (Notice that we also obtain an indirect proof of proposition 1 in this way.) □

Remark. The original statement of the Γ_{00} conjecture can also be reformulated in terms of the theta function. Indeed, if the matrix of coefficients of linear dependency is of rank N, i.e., if we have

$$c_{ij} = \sum_{n=1}^{N} U_i^{(n)} V_j^{(n)} + V_i^{(n)} U_j^{(n)},$$

then the functions τ and ψ would be obtained similarly, with

$$\tau(x_1, y_1, \dots, x_n, y_n, t) := \theta\left(\sum_{n=1}^{N} U^{(n)} x_n + V^{(n)} y_n + Pt + Z\right),$$

instead of (3) and ψ still given by (5), while u will take the form

$$u(x, y, t) := 2\sum_{n=1}^{N} \partial_{x_n}\partial_{y_n} \ln \tau(x_1, y_1, \dots, x_n, y_n, t),$$

instead of (4), and equation (6) from the main theorem will then still be satisfied. However, in this case the Taylor expansion for u in terms of x_1 near $x_1 = \eta(y_1, \dots, x_n, y_n, t)$ would contain a term

$$\sum_{n=2}^{N} \frac{\eta_{x_n y_n}}{x_1 - \eta},$$

of order -1, which would result in the computations to completely collapse. At the moment we do not see any way to extend our approach to prove the original Γ_{00}. Note that when thinking of Γ_{00} as a limiting case of semidegenerate trisecants, following proposition 1, we only get our special case, and thus potentially there could be abelian varieties that are not Jacobians violating the original Γ_{00} conjecture (though of course at the moment we are not aware of any such examples or candidates).

Note that for $g = 4$ the matrix of coefficients c_{ij} of the linear dependence for the two extra points $\pm(a - a')$, where a, a' are the g^1_3's on the curve (see [21]), is of rank greater than two.

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Part IV

Computation in Algebraic Geometry

This part has been coordinated by María Emilia Alonso with the advice of Laureano González-Vega and it is devoted to computation in algebraic geometry. It contains a short presentation, an unpublished article of Federico Gaeta and a research article.

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Federico Gaeta: His Last Ten Years of Mathematical Activity

María Emilia Alonso García

In his last ten years of mathematical activity, i.e., from 1990 to 2000, Federico Gaeta focused on Effective Methods in Algebraic Geometry (mainly Computational Invariant Theory), Combinatorics and Commutative Algebra, although his contributions in these fields were never published in journals. At the age of 70, while he was emeritus professor at the UCM, he participated in the spring semester at the IHP on *Surfaces de Riemann et Fibrés Vectoriels* where, after discussions with D. Eisenbud he wrote the paper *A fully explicit resolution of the ideal defining N generic points in the plane*. It is profusely quoted by many authors as a preprint of 1995. This paper was in fact a more explicit and computational version of an old publication of his, *Sur la distribution des degrés de formes appartenant à la matrice de l'ideal homogène attaché à un groupe de N points génériques du plan*, appeared in CR. Acad. Sci. Paris in 1951.

At the age of 73 he was awarded a fellowship (by the Fundación El Amo of the UCM) to visit the MSRI and the Department of Mathematics at Berkeley, to attend a semester about Combinatorics. There he met R. Stanley (Chair of the semester) and B. Sturmfels. He kept in contact with Bernd Sturmfels for some years, particularly when the latter was advisor for the PhD thesis of J.P. Dalbec (*Geometry and Combinatorics of Chow Forms*, 1996). Gaeta wrote a manuscript, *Grassmannians, Associated Forms to Cycles in* $\mathbb{P}^n$*, Chow manifolds*, quoted in Dalbec's thesis. He tried to turn this manuscript and the ideas of the paper he had published in Banach Centre (1990), into a book. He wrote hundreds of pages but unfortunately this book was never finished.

Coming back to Madrid he enrolled us (M.E. Alonso, L. González-Vega and later E. Briand) in a seminar whose aim was to revisit the Algebraic Invariant and Representation Theory from a computational and combinatorial point of view. We immediately noticed his vast knowledge in Algebraic Geometry and his quick and deep understanding of the problems. Unfortunately his old language made his arguments difficult to follow. However, as E. Briand himself explains in this

volume, his work has been greatly influenced by the ideas that Federico Gaeta shared in the seminar. At the meeting of the RSME (Royal Spanish Mathematical Society) held in Madrid, in 2000 we presented a poster titled *A computational approach to symbolic calculus in Invariant Theory* .

On the nineties he attended frequently the international conferences on Computational Algebra: MEGA 92, (Nice, France), MEGA 94 (Santander, Spain) and MEGA 96 (Eindhoven, Holland), and also some of the Spanish conferences EACA (Spanish National Conferences on Computer Algebra). He presented some communications, like for example *New non-recursive formulas for irreducible representations of* $GL(\mathbb{C}^{n+1})$ *and systems of equations for the symmetric powers* $Sym^n \mathbb{P}_m$, whose abstract is published in the SIGSAM Bulletin ACM, 1999.

In 1998, E. Arrondo, R. Mallavibarrena and myself organized a meeting on occasion of his 75th anniversary that was attended by many algebraic geometers of several countries; some of them have contributed to this volume. Most of the participants were from Italy "mi segunda patria", as he used to say.

His interest for Combinatorics came along with his friendship with Gian-Carlo Rota. They were introduced to each other, when Rota was a professor at Rockefeller University, by María Whonenburger, a brilliant Spanish mathematician that was professor at different American universities (cf. *Obituario por Gian Carlo Rota*, Gaceta de la RSME, 1999). Gaeta admired the idea of G.C. Rota of applying ideas coming from Algebraic Geometry to Combinatorics.

G.C. Rota, in *Ten Mathematics Problems I will never solve*[1] proposed a problem (Problem number 8) about *Confluent symmetric functions*. As Rota said, "Federico Gaeta, who is the last surviving student of Severi, has recently developed an as yet unpublished geometric theory of symmetric functions: the algebra of confluent symmetric functions fits into Gaeta's theory like a shoe". Since this work on geometric theory of symmetric functions was never published, we decided to do so in this volume, as a tribute to Federico Gaeta.

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[1]Invited address at the joint meeting of the American Mathematical Society and the Mexican Society, Oaxaca, Mexico December 6, 1997.

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Covariants Vanishing on Totally Decomposable Forms

Emmanuel Briand

In honor of Professor Federico Gaeta

Abstract. We consider the problem of providing systems of equations characterizing the forms with complex coefficients that are *totally decomposable*, i.e., products of linear forms. Our focus is computational. We present the well-known solution given at the end of the nineteenth century by Brill and Gordan and give a complete proof that their system does vanish only on the decomposable forms. We explore an idea due to Federico Gaeta which leads to an alternative system of equations, vanishing on the totally decomposable forms and on the forms admitting a multiple factor. Last, we give some insight on how to compute efficiently these systems of equations and point out possible further improvements.

Mathematics Subject Classification (2000). 15A72, 14Q15.

Keywords. Totally decomposable forms, Brills equations.

1. Introduction

Consider the following problem: find "good" systems of algebraic equations that characterize those forms with complex coefficients that are "totally decomposable", that is, products of linear forms. A solution of this problem was provided by Brill and Gordan [9] at the end of the nineteenth century. Their system of equations is obtained from a covariant (afterwards referred to as *Brill's covariant*) built from simple geometric considerations and classical constructions of the invariant theory of their time. The overall construction is, nevertheless, quite intricate and the systems of equations obtained this way are huge.

I met this problem several years ago when working on my thesis about the diagonal invariants of the symmetric group (a generalization of the symmetric

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functions arising naturally in some problems related to systems of polynomial equations). In 1999, Laureano Gonzalez Vega, one my thesis advisors, and MariEmi Alonso introduced me to Professor Gaeta. Besides marvelling me with the elegance of the algebraic geometry *à l'italienne*, he gave me some interesting suggestions. He designed a particularly simple alternative to Brill's covariant, another covariant that he called the *tangential*. The tangential of a form f is proportional to f if and only if f is totally decomposable or has a multiple factor. Nevertheless my work took a different direction and I didn't dwell on Gaeta's suggestions.

This paper presents in detail the construction of Gaeta's tangential and provides a formal proof of its fundamental property (Section 3). It also introduces a slight improvement: a new covariant (called *Gaeta's covariant*), built from the tangential, and which vanishes if and only if f is totally decomposable or has a multiple factor. The construction of Brill's covariant is also recalled, as well as the proof that Brill's covariant vanishes if and only if the form f is totally decomposable (Section 4). Note that Gordan's proof in [9] was not complete since it skipped the case when f had only multiple factors. The proof given here is complete. Last, Section 5 explains how to compute efficiently both Brill's and Gaeta's covariants and suggests some further improvements.

2. Preliminaries and main results

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let n and N be positive integers. We consider the forms of degree n with coefficients in $\mathbb{K}$ in the variables $x_1, x_2, \ldots, x_N$:

$$f(\mathbf{x}) = f(x_1, \ldots, x_N) = \sum a_\omega \mathbf{x}^\omega, \tag{1}$$

where the sum at the right-hand side is carried over all $\omega \in \mathbb{N}^N$ whose terms add to n, and $\mathbf{x}^\omega$ stands for $x_1^{\omega_1} x_2^{\omega_2} \cdots x_N^{\omega_N}$. Among these forms, some decompose totally, i.e., as products of linear forms. The set of the products of linear forms is an irreducible algebraic subvariety that we denote by $\mathcal{D}_n(\mathbb{K}^N)$, and call the *subvariety of totally decomposable forms*.

Problem. *Find a system of equations defining set-theoretically* $\mathcal{D}_n(\mathbb{K}^N)$.

Let us motivate this problem with an application. If one considers a form with rational, or even algebraic, coefficients, and wants to determine whether or not it decomposes totally, the most efficient way to do it is by applying an algorithm of absolute factorization. In this case, there is no hope that a system of equation for $\mathcal{D}_n(\mathbb{K}^N)$ would provide a better solution. A different problem is the following: given a family of forms $f_{t_1,t_2,\ldots}$ depending polynomially on parameters $t_1, t_2, \ldots$, determine for which values of the parameters the form is totally decomposable. Such a problem appears for instance in [11]. A system of equations defining the subvariety of the totally decomposable forms provides, by specialization, a system of equations in the parameters $t_1, t_2, \ldots$ whose solutions are the solutions of the original problem.

Brill [3] and Gordan [9] found a system of equations defining $\mathcal{D}_n(\mathbb{K}^N)$ set-theoretically. These equations are called *Brill's equations* in [8], and the name was used afterwards by other authors. Brill's equations are obtained as the coefficients of *Brill's covariant*, a polynomial function $B : f \mapsto B_f$ from the space of forms of degree n over $\mathbb{K}^N$ to the space of polynomials in the $3N$ variables from the following three families:

$$\mathbf{x} = (x_1, \ldots, x_N), \qquad \mathbf{y} = (y_1, \ldots, y_N), \qquad \mathbf{z} = (z_1, \ldots, z_N).$$

So for each f, the polynomial B_f writes

$$B_f = \sum b_{\alpha,\beta,\gamma}(f)\mathbf{x}^\alpha \mathbf{y}^\beta \mathbf{z}^\gamma,$$

where the indices α, β, γ are in $\mathbb{N}^N$. Its coefficients $b_{\alpha,\beta,\gamma}(f)$ are polynomial functions of f, that is polynomials in the indeterminate coefficients a_ω of f as in (1). Brill's equations are the equations:

$$b_{\alpha,\beta,\gamma}(f) = 0.$$

Theorem 1. *Let f be a form with coefficients in $\mathbb{K}$. Then f is totally decomposable if and only if B_f vanishes identically with respect to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.*

Otherwise stated, Brill's equations define $\mathcal{D}_n(\mathbb{K}^N)$ set-theoretically.

The name *Brill's covariant* refers to the following equivariance property of the map B. Let $V = \mathbb{K}^N$. The target space and source space of B are representations of $SL(V)$: the source space, the space of forms of degree n on V, is the symmetric power S^nV^*. The target space $\mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is the symmetric algebra $S^\bullet(V^* \oplus V^* \oplus V^*)$ on three copies of the dual of V. This space can be identified with the space of polynomial functions on $V \oplus V \oplus V$. The group $SL(V)$ acts diagonally on this direct sum. This means that for all $M \in SL(V)$ and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V,

$$M(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (M(\mathbf{u}), M(\mathbf{v}), M(\mathbf{w})).$$

With this interpretation, we have for all $f \in S^nV^*$ and all $M \in SL(V)$:

$$B_{f \circ M} = B_f \circ M.$$

Brill's covariant has another property of invariance, with respect to the group SL_2, that acts on the first two copies of V in the direct sum $V \oplus V \oplus V$:

$$V \oplus V \oplus V \cong (V \oplus V) \oplus V \cong (V \otimes \mathbb{K}^2) \oplus V.$$

The action of SL_2 is as follows: for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all $\theta = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2$,

$$\theta(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (a\mathbf{u} + b\mathbf{v}, c\mathbf{u} + d\mathbf{v}, \mathbf{w}). \tag{2}$$

We have for all $f \in S^nV^*$ and all $\theta \in SL_2$:

$$B_f \circ \theta = B_f.$$

The invariance property with respect to SL_2 implies that for all form f, the polynomial B_f can be written as a polynomial in the variables $\mathbf{z}$ and in the brackets $[i, j] = x_iy_j - x_jy_i$ for $1 \leq i < j \leq N$. See Section 5.

Let $\mathcal{A}$ be the ring of the $SL(V)$-equivariant polynomial maps from S^nV^* to $S^\bullet(V\oplus V\oplus V)^*$ that are invariant under SL_2 for the above action (2). After what precedes, the map B belongs to $\mathcal{A}$. The ring $\mathcal{A}$ is endowed with a grading with values in $\mathbb{N}^3$:

$$\begin{aligned}\mathcal{A} &\cong \left(S^\bullet\left(S^nV\right)\otimes S^\bullet(V\oplus V\oplus V)^*\right)^{SL(V)\times SL_2}\\ &\cong \bigoplus_{(d,j,k)\in\mathbb{N}^3}\left(S^dS^nV\otimes\left(S^jV^*\otimes S^jV^*\right)^{SL_2}\otimes S^kV^*\right)^{SL(V)}.\end{aligned}$$

We will refer to this grading as *multidegree*. The homogeneous elements of $\mathcal{A}$ of multidegree (d,j,k) are the elements of $\mathcal{A}$ that are homogeneous of degree d in the coefficients of f, homogeneous of degree j in the variables $\mathbf{x}$ as well as in the variables $\mathbf{y}$, and homogeneous of degree k in the variables $\mathbf{z}$.

It turns out that Brill's covariant is homogeneous of multidegree $(n+1,n,n^2-n)$.

Gaeta designed another homogeneous element of $\mathcal{A}$, the *tangential*, $T: f\mapsto T_f$ with the following properties.

Theorem 2 (Gaeta). *Let f be a form of degree n on V. Then:*

- *For all $\mathbf{u}$ and $\mathbf{v}$ in V, the form $T_f(\mathbf{u},\mathbf{v},\mathbf{z})$ (in the variables $\mathbf{z}$) is either 0 or totally decomposable.*
- *The form f is totally decomposable or admits a multiple factor if and only if T_f is a multiple of $f(\mathbf{z})$ in the ring of polynomials in $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.*
- *The form f admits a multiple factor if and only if T_f vanishes identically.*

That the form f has a multiple factor means that it factorizes into gh^2 with g and h forms and h non-constant.

Gaeta's tangential is provided by a simple formula: $G_f(\mathbf{x},\mathbf{y},\mathbf{z})$ is the Sylvester resultant of two binary forms in (λ,μ): the form f evaluated at $\lambda\mathbf{x}+\mu\mathbf{y}$, and the directional derivative $D_{\mathbf{z}}f$ of f in the direction $\mathbf{z}$, taken at the point $\lambda\mathbf{x}+\mu\mathbf{y}$. The tangential T is a homogeneous element of $\mathcal{A}$ with multidegree $(2n-1,n^2-n,n)$.

In this paper we improve Gaeta's construction as follows: we build from T a new map $G\in\mathcal{A}$ (homogeneous of the same multidegree as T) with the property:

Theorem 3. *Let f be a form of degree n on V. Then f is totally decomposable or admits a multiple factor if and only if G_f vanishes identically with respect to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.*

We refer to the new polynomial G_f as *Gaeta's covariant*. Its construction, which is simpler than the construction of Brill's covariant, is presented in Section 3. The construction of Brill's covariant is also recalled in Section 4.

3. Gaeta's tangential and Gaeta's covariant

In this section we introduce Gaeta's tangential and Gaeta's covariant. A preliminary about classical objects from invariant theory, the *polars* of a form, is needed.

3.1. The polars of a form f

Let f be a form of degree n in N variables. Let $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ be two families of variables. Consider the decomposition:

$$f(\mathbf{x}+\mathbf{y}) = \sum_{i=0}^{n} f^{(i)}(\mathbf{x};\mathbf{y}),$$

where $f^{(i)}(x;y)$ is the component of degree i in the x-variables (and thus equivalently: of degree $n-i$ in the y-variables). The bihomogeneous forms $f^{(i)}(\mathbf{x};\mathbf{y})$ are the *polars*[1] *of* f.

Let $\mathbf{u}$ and $\mathbf{v}$ in V with $\mathbf{v}$ non-zero. Then $f^{(1)}(\mathbf{v};\mathbf{u})$ is the directional derivative of f at $\mathbf{u}$, in the direction $\mathbf{v}$:

$$f^{(1)}(\mathbf{v};\mathbf{u}) = D_{\mathbf{v}} f(\mathbf{u}).$$

In particular, for $\mathbf{u}$ non-zero, the equation $f^{(1)}(\mathbf{z},\mathbf{u}) = 0$ is an equation (in $\mathbf{z}$) of the tangent space at $[\mathbf{u}]$ for the hypersurface $f = 0$ of $\mathbb{P}(V)$.

We conclude with an observation about the polars of a decomposable form. Let e_1, e_2, …, e_n be the elementary symmetric polynomials in n variables.

Lemma 4. *Let f be a form of degree n. Assume that f is totally decomposable: $f = \ell_1\ell_2\cdots\ell_n$ with ℓ_1, ℓ_2, …, ℓ_n linear.*

Then for all i between 0 and n, the polar $f^{(i)}(\mathbf{x};\mathbf{y})$ is the following homogenized elementary symmetric polynomial:

$$f^{(i)}(\mathbf{x};\mathbf{y}) = e_i\left(\frac{\ell_1(\mathbf{x})}{\ell_1(\mathbf{y})}, \frac{\ell_2(\mathbf{x})}{\ell_2(\mathbf{y})}, \ldots, \frac{\ell_n(\mathbf{x})}{\ell_n(\mathbf{y})}\right)\cdot f(\mathbf{y}) = \sum_{I\subset[n],\#I=i}\ \prod_{k\in I}\ell_k(\mathbf{x})\prod_{k\notin I}\ell_k(\mathbf{y}).$$

3.2. Gaeta's tangential

Gaeta [7] proposed the following geometric construction to detect totally decomposable forms. Its setting is the projective space $\mathbb{P}(V)$ (recall that $V = \mathbb{K}^N$). Given a non-zero vector $\mathbf{u} \in V$, the corresponding point in $\mathbb{P}(V)$ will be denoted with $[\mathbf{u}]$.

Let f be a form of degree n on V. Assume that f has no multiple factor. A generic line L in $\mathbb{P}(V)$ meets the hypersurface H_f of $\mathbb{P}(V)$ defined by $f = 0$ at n distinct points, all non-singular on H_f. Then *f is totally decomposable if and only if H_f coincides with the union $U(L,f)$ of the tangent hyperplanes of H_f at these n intersections.*

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors such that $[\mathbf{u}]$ and $[\mathbf{v}]$ span L, then $U(L,f)$ is the zero locus of

$$\prod_{(\lambda:\mu)\in\mathbb{P}^1 \mid f(\lambda\,\mathbf{u}+\mu\,\mathbf{v})=0} f^{(1)}(\mathbf{z};\lambda\,\mathbf{u}+\mu\,\mathbf{v}). \tag{3}$$

One recognizes in this expression the Sylvester resultant of $f(\lambda\,\mathbf{u}+\mu\,\mathbf{v})$ and $f^{(1)}(\mathbf{z};\lambda\,\mathbf{u}+\mu\,\mathbf{v})$, as binary forms in (λ,μ).

[1] Some scalar factor may be needed to fit in with the classical notations.

For any $f \in S^n V^*$, Gaeta's tangential T_f is defined as the following polynomial in $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$:

$$T_f = \mathsf{Resultant}_{(\lambda,\mu)} \left(f(\lambda\,\mathbf{x} + \mu\,\mathbf{y}), f^{(1)}(\mathbf{z}; \lambda\,\mathbf{x} + \mu\,\mathbf{y}) \right). \tag{4}$$

Observe that T is a homogeneous element of $\mathcal{A}$ of multidegree $(2\,n-1, n^2-n, n)$.

Example 1. *Consider the case of quadratic forms. Let f be a quadratic form in N variables. We have:*

$$f(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2\ f(\mathbf{x}) + \lambda\mu\ f^{(1)}(\mathbf{x}; \mathbf{y}) + \mu^2\ f(\mathbf{y}),$$
$$f^{(1)}(\mathbf{z}; \lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\ f^{(1)}(\mathbf{x}; \mathbf{z}) + \mu\ f^{(1)}(\mathbf{y}; \mathbf{z}).$$

Therefore,

$$T_f = \begin{vmatrix} f(\mathbf{x}) & f^{(1)}(\mathbf{x}; \mathbf{y}) & f(\mathbf{y}) \\ f^{(1)}(\mathbf{x}; \mathbf{z}) & f^{(1)}(\mathbf{y}; \mathbf{z}) & 0 \\ 0 & f^{(1)}(\mathbf{x}; \mathbf{z}) & f^{(1)}(\mathbf{y}; \mathbf{z}) \end{vmatrix}$$

We now proceed to the proof of Theorem 2.

Proof of Theorem 2. Let f be a form of degree n on V.

That $T_f(\mathbf{u}, \mathbf{v}, \mathbf{z})$ is either 0 or totally decomposable for any choice of $\mathbf{u}$ and $\mathbf{v}$ follows from the factorization (3) up to a factor depending only on $\mathbf{u}$ and $\mathbf{v}$.

Consider the assertions:

(i) *The form f has a multiple factor.*
(ii) *The form T_f vanishes identically.*
(iii) *The form f is totally decomposable or admits a multiple factor.*
(iv) *The form T_f is a multiple of $f(\mathbf{z})$.*

We should establish on the one hand, the equivalence of (i) and (ii), and, on the other hand, the equivalence of (iii) and (iv).

Let us first establish that (i) and (ii) are equivalent.

Suppose that (i) holds: $f = gh$ with g non-constant and g divides h. After the interpretation of the first polar as a directional derivative (see Section 3.1) and Leibnitz' rule for the derivative of a product, we have:

$$f^{(1)}(\mathbf{z}, \mathbf{x}) = g(\mathbf{x})\, h^{(1)}(\mathbf{z}; \mathbf{x}) + h(\mathbf{x})\, g^{(1)}(\mathbf{z}; \mathbf{x}).$$

In particular $g(\mathbf{x})$ divides also $f^{(1)}(\mathbf{z}; \mathbf{x})$. Therefore the binary form $g(\lambda\mathbf{x} + \mu\mathbf{y})$ is a non-trivial common factor of $f(\lambda\mathbf{x} + \mu\mathbf{y})$ and $f^{(1)}(\mathbf{z}; \lambda\mathbf{x} + \mu\mathbf{y})$. We conclude that $T_f = 0$ after (4), i.e., (ii) holds.

Suppose now that (i) does not hold. Then there exists a line L cutting H_f at n distinct points. Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors of V such that L is the line joining $[\mathbf{u}]$ and $[\mathbf{v}]$, and $[\mathbf{u}]$ does not belong to H_f. Then $T_f(\mathbf{u}, \mathbf{v}, \mathbf{u})$ is, up to a non-zero scalar, the discriminant of the binary form $f(\lambda\mathbf{u} + \mu\mathbf{v})$, which has n distinct roots. Therefore this discriminant is non-zero. Since T_f does not vanish at $(\mathbf{u}, \mathbf{v}, \mathbf{u})$, it does not vanish identically, i.e., (ii) does not hold.

Let us now establish the equivalence of (iii) and (iv).

Suppose that (iii) holds. If f admits a multiple factor, then $T_f = 0$, which is a multiple of $f(\mathbf{z})$. In this case (iv) holds. It remains to show that (iv) also holds when f has no multiple factor and is totally decomposable. This is implied straightforwardly by the following property:

(v) *If ℓ is a linear factor of f, then $\ell(\mathbf{z})$ is also a factor of T_f.*

Let us prove (v). Assume that $f = \ell\, h$ with ℓ linear. After Leibnitz' rule:

$$f^{(1)}(\mathbf{z};\mathbf{x}) = \ell(\mathbf{x})h^{(1)}(\mathbf{z};\mathbf{x}) + \ell(\mathbf{z})h(\mathbf{x}).$$

Let $\mathbf{w}$ be a vector of V such that $\ell(\mathbf{w}) = 0$. Then the binary forms $f(\lambda\mathbf{x} + \mu\mathbf{y})$ and $f^{(1)}(\mathbf{w};\lambda\mathbf{x} + \mu\mathbf{y})$ have the common factor $\ell(\lambda\mathbf{x} + \mu\mathbf{y})$. It follows that their resultant $T_f(\mathbf{x},\mathbf{y},\mathbf{w})$ is zero. This shows that the zero locus of $\ell(\mathbf{z})$ is contained in the zero locus of $T_f(\mathbf{x},\mathbf{y},\mathbf{z})$. Since $\ell(\mathbf{z})$ is irreducible, we conclude that it divides $T_f(\mathbf{x},\mathbf{y},\mathbf{z})$, as required.

Assume now that (iv) holds: there exists a polynomial $R(\mathbf{x},\mathbf{y},\mathbf{z})$ such that $T_f(\mathbf{x},\mathbf{y},\mathbf{z}) = R(\mathbf{x},\mathbf{y},\mathbf{z}) \cdot f(\mathbf{z})$. Since T_f and $f(\mathbf{z})$ are both homogeneous of degree n in $\mathbf{z}$, the polynomial R has degree 0 in $\mathbf{z}$ and we can write $R = R(\mathbf{x},\mathbf{y})$. If $R = 0$ then $T_f = 0$. Then f admits a multiple factor. If $R \neq 0$ then there exist vectors $\mathbf{u}$ and $\mathbf{v}$ in V such that $R(\mathbf{u},\mathbf{v}) \neq 0$. We have:

$$T_f(\mathbf{u},\mathbf{v},\mathbf{z}) = R(\mathbf{u},\mathbf{v}) \cdot f(\mathbf{z}).$$

Since $T_f(\mathbf{u},\mathbf{v},\mathbf{z})$ is totally decomposable, so is $f(\mathbf{z})$, i.e., (iii) holds. □

The properties of invariance with respect to SL_2 and covariance with respect to $SL(V)$ of T_f follow easily from (4).

That T_f is proportional to $f(\mathbf{z})$ could provide a system of equations characterizing the totally decomposable forms (among the forms with no multiple factor) as follows. Decompose:

$$f(\mathbf{z}) = \sum_{\omega} a_\omega \mathbf{z}^\omega, \quad T_f = \sum_{\omega} t_\omega(f,\mathbf{x},\mathbf{y})\mathbf{z}^\omega.$$

The forms are proportional (as forms in $\mathbf{z}$) if and only if all determinants $\left|\begin{smallmatrix} a_\alpha & a_\beta \\ t_\alpha & t_\beta \end{smallmatrix}\right|$ vanish identically with respect to $\mathbf{x}$ and $\mathbf{y}$.

We present in what follows how to obtain equations of smaller degree.

3.3. Gaeta's covariant

Let us consider the ratio $\delta_f = T_f / f(\mathbf{z})$. When f is totally decomposable, this ratio δ_f admits a simple expression in term of the linear factors of f. We will first derive this expression and next exhibit a polynomial covariant continuation Δ for δ.

Remember that for any homogeneous binary form $\psi(\lambda,\mu)$ and any family of numbers $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$:

$$\mathsf{Resultant}_{(\lambda,\mu)}\left(\prod_{i=1}^{n}(a_i\lambda + b_i\mu);\ \psi(\lambda,\mu)\right) = \prod_{i=1}^{n}\psi(-b_i, a_i).$$

This shows that when f decomposes totally as $\ell_1\ell_2\cdots\ell_n$,

$$\begin{aligned} T_f &= \prod_{i=1}^{n} f^{(1)}(\mathbf{z};\ -\ell_i(\mathbf{y})\mathbf{x} + \ell_i(\mathbf{x})\mathbf{y}) \\ &= \prod_{i=1}^{n}\sum_{k_i=1}^{n} \ell_{k_i}(\mathbf{z}) \prod_{j\neq k_i} (\ell_i(\mathbf{x})\ell_j(\mathbf{y}) - \ell_i(\mathbf{y})\ell_j(\mathbf{x})) \\ &= \prod_{i=1}^{n} \ell_i(\mathbf{z}) \prod_{j\neq i} (\ell_i(\mathbf{x})\ell_j(\mathbf{y}) - \ell_i(\mathbf{y})\ell_j(\mathbf{x})) \\ &= f(\mathbf{z}) \prod_{(i,j)\,|\,i\neq j} \begin{vmatrix} \ell_i(\mathbf{x}) & \ell_j(\mathbf{x}) \\ \ell_i(\mathbf{y}) & \ell_j(\mathbf{y}) \end{vmatrix}. \end{aligned}$$

Therefore,

$$\delta_f = \prod_{(i,j)\,|\,i\neq j} \begin{vmatrix} \ell_i(\mathbf{x}) & \ell_j(\mathbf{x}) \\ \ell_i(\mathbf{y}) & \ell_j(\mathbf{y}) \end{vmatrix}.$$

We recognize a discriminant. Given a binary form ϕ of degree n in the variables (λ, μ):

$$\phi(\lambda,\mu) = c_n\lambda^n + c_{n-1}\lambda^{n-1}\mu + \cdots = \prod_{i=1}^{n}(\lambda a_i + \mu b_i),$$

its discriminant is defined as:

$$\mathsf{Disc}_{(\lambda,\mu)}(\phi) = \mathsf{Resultant}_{(\lambda,\mu)}\left(\phi; \frac{\partial\phi}{d\lambda}\right)/c_n.$$

(Note that the sign may differ from other definitions.) It fulfills:

$$\mathsf{Disc}_{(\lambda,\mu)}(\phi) = \prod_{(i,j)\,|\,i\neq j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}.$$

Therefore, for f totally decomposable,

$$\delta_f = \mathsf{Disc}_{(\lambda,\mu)}\left(f(\lambda\,\mathbf{x} + \mu\,\mathbf{y})\right).$$

We now set Δ_f for $\mathsf{Disc}_{(\lambda,\mu)}(f(\lambda\,\mathbf{x} + \mu\,\mathbf{y}))$ and we define:

$$G_f = T_f - \Delta_f \cdot f(\mathbf{z}). \tag{5}$$

The polynomial map Δ is clearly equivariant under $SL(V)$ and invariant under SL_2. So is G, after the above equality (5). Theorem 3 follows from the above computations and Theorem 2.

We conclude this section with a remark: for $\phi(\lambda,\mu) = f(\lambda\mathbf{x} + \mu\mathbf{y})$, we have

$$\frac{\partial\phi}{d\,\lambda} = f^{(1)}(\mathbf{x}; \lambda\mathbf{x} + \mu\mathbf{y}).$$

Moreover, the coefficient of λ^n in ϕ is $f(\mathbf{x})$. Therefore

$$\Delta_f = \frac{1}{f(\mathbf{x})}\ \mathsf{Resultant}_{(\lambda,\mu)}\left(f(\lambda\mathbf{x} + \mu\mathbf{y}); f^{(1)}(\mathbf{x}; \lambda\mathbf{x} + \mu\mathbf{y})\right).$$

Comparing with (4) we obtain that:

$$\Delta_f = T_f(\mathbf{x}, \mathbf{y}, \mathbf{x})/f(\mathbf{x}).$$

Example 2 (Continuation of Example 1). *For f quadratic we have:*

$$T_f = \begin{vmatrix} f(\mathbf{x}) & f^{(1)}(\mathbf{x};\mathbf{y}) & f(\mathbf{y}) \\ f^{(1)}(\mathbf{x};\mathbf{z}) & f^{(1)}(\mathbf{y};\mathbf{z}) & 0 \\ 0 & f^{(1)}(\mathbf{x};\mathbf{z}) & f^{(1)}(\mathbf{y};\mathbf{z}) \end{vmatrix}$$

The covariant Δ_f is obtained by replacing $\mathbf{z}$ with $\mathbf{x}$ in T_f, and dividing by $f(\mathbf{x})$. For f quadratic we have $f^{(1)}(\mathbf{x};\mathbf{x}) = 2\,f(\mathbf{x})$. We obtain:

$$\Delta_f = 4\,f(\mathbf{x})f(\mathbf{y}) - \left(f^{(1)}(\mathbf{x};\mathbf{y})\right)^2.$$

Expanding $T_f - \Delta_f \cdot f(\mathbf{z})$ we get:

$$G_f = f(\mathbf{x})\left(f^{(1)}(\mathbf{y};\mathbf{z})\right)^2 + f(\mathbf{y})\left(f^{(1)}(\mathbf{x};\mathbf{z})\right)^2 - f^{(1)}(\mathbf{x};\mathbf{y})\; f^{(1)}(\mathbf{x};\mathbf{z})\; f^{(1)}(\mathbf{y};\mathbf{z}) - 4\,f(\mathbf{x})f(\mathbf{y})f(\mathbf{z}) + f(\mathbf{z})\left(f^{(1)}(\mathbf{x};\mathbf{y})\right)^2.$$

This can be presented as follows:

$$G_f = -4 \begin{vmatrix} f(\mathbf{x}) & f^{(1)}(\mathbf{x};\mathbf{y})/2 & f^{(1)}(\mathbf{x};\mathbf{z})/2 \\ f^{(1)}(\mathbf{x};\mathbf{y})/2 & f(\mathbf{y}) & f^{(1)}(\mathbf{y};\mathbf{z})/2 \\ f^{(1)}(\mathbf{x};\mathbf{z})/2 & f^{(1)}(\mathbf{y};\mathbf{z})/2 & f(\mathbf{z}) \end{vmatrix}$$

4. Brill's covariant

In this section we report on Gordan's geometric presentation [9] of Brill's covariant. Such a presentation is already provided by the monograph [8], and of course by Gordan's original text [9], but in neither case the proof of Theorem 1 is complete.

As a preliminary, a classical construction in invariant theory, needed in the sequel, is presented.

4.1. The apolar covariant

Consider a pair of binary forms of degree n:

$$\phi(\lambda,\mu) = \sum \phi_{i,j}\lambda^i\mu^j, \qquad \psi(\lambda,\mu) = \sum \psi_{i,j}\lambda^i\mu^j.$$

Their *apolar form* is the scalar:

$$\begin{aligned} \mathsf{Apo}_{(\lambda,\mu)}(\phi,\psi) &= \phi\left(-\frac{\partial}{d\,\mu}, \frac{\partial}{d\,\lambda}\right)\; \psi(\lambda,\mu), \\ &= \sum_{i+j=n} (-1)^i i!j!\phi_{i,j}\psi_{j,i}. \end{aligned}$$

One says that ϕ and ψ are apolar to each other if their apolar form is zero. The apolar form is a *combinant*. This means that it is an invariant under the two actions of SL_2:

- Under change of variables: for all $\theta \in SL_2$ and all binary forms ϕ and ψ of degree n,
$$\mathsf{Apo}_{(\lambda,\mu)}(\phi \circ \theta, \psi \circ \theta) = \mathsf{Apo}_{(\lambda,\mu)}(\phi, \psi).$$
- Under linear combinations of the two quadratic forms: for all $\theta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2$ and all binary forms ϕ and ψ of degree n,
$$\mathsf{Apo}_{(\lambda,\mu)}(a\phi + b\psi, c\phi + d\psi) = \mathsf{Apo}_{(\lambda,\mu)}(\phi, \psi).$$

The apolar form was very well known at the end of the nineteenth century. The following properties were familiar at the time.

Lemma 5. *Let ϕ be a binary form of degree n.*

- *Let $\ell_1\ell_2\cdots\ell_n$ be a factorization of ϕ as a product of linear forms. Then all the linear combinations of ℓ_1^n, ℓ_2^n, …, ℓ_n^n are apolar to ϕ.*
- *Let ℓ be a linear form. Assume that ℓ^n is apolar to ϕ. Then ℓ divides ϕ.*

Clebsch's transfer principle is a classical method to produce from each invariant $I_{(\lambda,\mu)}$ of pairs of binary forms in (λ, μ) of degree n a covariant $C_{I,\mathbf{x}}$ of pairs of forms in $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ of degree n:
$$C_{I,\mathbf{x}}(f, g)(\mathbf{x}, \mathbf{y}) = I_{(\lambda,\mu)}(f(\lambda\mathbf{x} + \mu\mathbf{y}), g(\lambda\mathbf{x} + \mu\mathbf{y})).$$
If $I_{(\lambda,\mu)}$ is a combinant then $C_{I,\mathbf{x}}$ is not only invariant under SL_N acting by changes of variables, but also under SL_2 acting by linear combination of the families of variables $\mathbf{x}$ and $\mathbf{y}$. That is, $C_{I,\mathbf{x}}$ belongs to the algebra $\mathcal{A}$ introduced in Section 2.

Applying Clebsch's transfer principle to the apolar form produces the *apolar covariant* of pairs (f, g) of N-ary forms of degree n:
$$C_{\mathsf{Apo},\mathbf{x}}(f, g)(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} (-1)^k k!(n-k)! f^{(k)}(\mathbf{x}; \mathbf{y}) g^{(n-k)}(\mathbf{x}; \mathbf{y}).$$
As in the case of binary forms, we will say that the N-ary forms f and g are *apolar* when their apolar covariant vanishes.

In the proof of Proposition 2.10 of [8], the apolar covariant is interpreted in the language of modern representation theory. It corresponds, up to a scalar factor, to the projection from $S^n(V^*) \otimes S^n(V^*)$ to the Weyl module $S^{(n,n)}(V^*)$ in the decomposition
$$S^n(V^*) \otimes S^n(V^*) = \bigoplus_{k=0}^{n} S^{(n+k,n-k)}(V^*).$$
and is called there *Young's vertical multiplication*.

Suppose that f and g are forms of degree n on $V = \mathbb{K}^N$ and $\mathbf{u}$ and $\mathbf{v}$ are non-proportional vectors in V. Then f and g restrict to binary forms on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. That $C_{\mathsf{Apo},\mathbf{x}}(f, g)(\mathbf{u}, \mathbf{v})$ vanishes means that these restrictions are apolar. The following properties of the apolar covariant are straightforwardly deduced from this remark and the corresponding properties of the apolar form (Lemma 5).

Lemma 6. *Let f be a form of degree n on V.*

- *Assume that f is totally decomposable, factorizing in a product of linear forms as $f = \ell_1\ell_2\cdots\ell_n$. Then, for all linear combinations g of ℓ_1^n, ℓ_2^n, ..., ℓ_n^n, the apolar covariant $C_{\mathsf{Apo},\mathbf{x}}(f,g)$ vanished identically.*
- *Let ℓ be a linear form on V. If $C_{\mathsf{Apo},\mathbf{x}}(f,\ell^n)$ vanishes identically then ℓ divides f.*

4.2. Gordan's presentation

Let f be a totally decomposable form of degree n that decomposes as a product of linear forms as $f = \ell_1\ell_2\cdots\ell_n$. The polars of f, after suitable normalization, will be interpreted as the elementary symmetric polynomials in scalar multiples of $\ell_1(\mathbf{x})$, $\ell_2(\mathbf{x})$, ..., $\ell_n(\mathbf{x})$. We will be able to compute the corresponding nth symmetric power sum, which, after Lemma 6, will be apolar to f. The vanishing of the apolar covariant of f and the nth symmetric power sum will therefore be a necessary condition for total decomposability. This condition will be shown to be also sufficient.

After Lemma 4, for all i between 0 and n we have:

$$f^{(i)}(\mathbf{x};\mathbf{z}) = f(\mathbf{z})\cdot e_i\left(\frac{\ell_1(\mathbf{x})}{\ell_1(\mathbf{z})};\cdots;\frac{\ell_n(\mathbf{x})}{\ell_n(\mathbf{z})}\right).$$

Let us clear all denominators by multiplying with $f(\mathbf{z})^{i-1}$:

$$f(\mathbf{z})^{i-1}f^{(i)}(\mathbf{x};\mathbf{z}) = e_i\left(\frac{f(\mathbf{z})}{\ell_1(\mathbf{z})}\ell_1(\mathbf{x});\cdots;\frac{f(\mathbf{z})}{\ell_n(\mathbf{z})}\ell_n(\mathbf{x})\right).$$

The polynomials $f(\mathbf{z})^{i-1}f^{(i)}(\mathbf{x};\mathbf{z})$ are therefore the elementary symmetric polynomials in the roots of

$$\sum_{i=0}^{n} f^{(i)}(\mathbf{x};\mathbf{z})f(\mathbf{z})^{i-1}(-1)^i t^{n-i} = f\left(t\,\mathbf{z} - f(\mathbf{z})\mathbf{x}\right)/f(\mathbf{z}). \tag{6}$$

The nth symmetric power sum p_n in n variables has a unique representation as a polynomial in the elementary symmetric polynomials. Let P_n be the polynomial such that

$$p_n = P_n(e_1, e_2, \ldots, e_n).$$

For any form f of degree $d \le n$ define:

$$Q_n(f) = P_n\left(f^{(1)}(\mathbf{x};\mathbf{z}),\ f(\mathbf{z})f^{(2)}(\mathbf{x};\mathbf{z}),\ldots,\ f(\mathbf{z})^{d-1}f^{(d)}(\mathbf{x};\mathbf{z}),0,\ldots,0\right).$$

Then $Q_n(f)$ is a covariant of the forms f. As a consequence, so is the apolar covariant of f and $Q_n(f)$. Brill's covariant is this polynomial:

$$B_f = C_{\mathsf{Apo},\mathbf{x}}(f, Q_n(f)).$$

Example 3. *For f quadratic we have:*

$$p_2 = e_1^2 - 2\,e_2.$$

Therefore,

$$Q_2(f) = \left(f^{(1)}(\mathbf{x};\mathbf{z})\right)^2 - 2\, f(\mathbf{x})f(\mathbf{z}).$$

Setting g for $Q_2(f)$ we have $g^{(0)}(\mathbf{x};\mathbf{y}) = g(\mathbf{y})$, $g^{(2)}(\mathbf{x};\mathbf{y}) = g(\mathbf{x})$ and:

$$g^{(1)}(\mathbf{x};\mathbf{y}) = 2\, \left(f^{(1)}(\mathbf{x};\mathbf{z})f^{(1)}(\mathbf{y};\mathbf{z}) - f^{(1)}(\mathbf{x};\mathbf{y})f(\mathbf{z})\right).$$

Therefore,

$$\begin{aligned} B_f = {} & 2\, f(\mathbf{x}) \left(\left(f^{(1)}(\mathbf{y};\mathbf{z})\right)^2 - 2\, f(\mathbf{y})f(\mathbf{z}) \right) \\ & - 2\, f^{(1)}(\mathbf{x};\mathbf{y}) \left(f^{(1)}(\mathbf{x};\mathbf{z})f^{(1)}(\mathbf{y};\mathbf{z}) - f^{(1)}(\mathbf{x};\mathbf{y})f(\mathbf{z})\right) \\ & + 2\, f(\mathbf{y}) \left(\left(f^{(1)}(\mathbf{x};\mathbf{z})\right)^2 - 2\, f(\mathbf{x})f(\mathbf{z}) \right) \end{aligned}$$

This can be presented as:

$$B_f = -\begin{vmatrix} 2\, f(\mathbf{x}) & f^{(1)}(\mathbf{x};\mathbf{y}) & f^{(1)}(\mathbf{x};\mathbf{z}) \\ f^{(1)}(\mathbf{x};\mathbf{y}) & 2\, f(\mathbf{y}) & f^{(1)}(\mathbf{y};\mathbf{z}) \\ f^{(1)}(\mathbf{x};\mathbf{z}) & f^{(1)}(\mathbf{y};\mathbf{z}) & 2\, f(\mathbf{z}) \end{vmatrix}$$

The covariant Q_n is a homogeneous element of the ring $\mathcal{A}$ (see Section 2) with multidegree $(n, n, n^2 - n)$. As a consequence, B is homogeneous of multidegree $(n+1, n, n^2 - n)$.

By construction, the identical vanishing of B_f is a necessary condition for total decomposability of f. Let us show now that the identical vanishing of B_f is also a sufficient condition for total decomposability of f. The proof rests on the following lemma.

Lemma 7. *Let f be a form of degree n on V whose Brill covariant vanishes identically. Let $[\mathbf{w}]$ be a smooth point on the reduced hypersurface H_f of $\mathbb{P}(V)$ defined by $f = 0$. Then the irreducible component of H_f containing $[\mathbf{w}]$ is a hyperplane.*

Observe that, under the hypotheses of the Lemma, there is a unique irreducible factor g of f such that $g(\mathbf{w}) = 0$, and the linear form $g(\mathbf{x};\mathbf{w})$ is non-zero.

Admit the lemma for now. Suppose that B_f vanishes identically. Any irreducible component Γ of H_f has a smooth point $[\mathbf{w}]$ not contained in any other irreducible component. After the lemma, Γ is necessarily an hyperplane. Therefore f is totally decomposable.

Theorem 1 will therefore be proved, as soon as we have proved Lemma 7.

Proof of Lemma 7. Let $g = 0$ be an equation for the irreducible component of H_f containing $[\mathbf{w}]$.

Consider first the simplest case: the factor g of f is not multiple. Since B_f vanishes identically, in particular $B_f(\mathbf{x}, \mathbf{y}, \mathbf{w})$ vanishes identically with respect to $\mathbf{x}$ and $\mathbf{y}$. The evaluation at $\mathbf{z} = \mathbf{w}$ of the polynomial defined in (6) is:

$$t^n - t^{n-1} f^{(1)}(\mathbf{x}, \mathbf{w}).$$

Remember that $f^{(1)}(\mathbf{x},\mathbf{w})=0$ is an equation (in $\mathbf{x}$) for the tangent space of f at $[\mathbf{w}]$. In particular, since $[\mathbf{w}]$ is non-singular on H_f and cancels no multiple factor of f, the linear form $f^{(1)}(\mathbf{x},\mathbf{w})$ is non-zero. Therefore, the roots of $f\,(t\mathbf{w}-f(\mathbf{w})\mathbf{x})$ are 0 (with multiplicity $n-1$) and $f^{(1)}(\mathbf{x},\mathbf{w})$ (with multiplicity 1). Thus the corresponding nth symmetric power sum is $Q_n(f)(\mathbf{x},\mathbf{w}) = \left(f^{(1)}(\mathbf{x},\mathbf{w})\right)^n$. It is apolar to f since $B_f(\mathbf{x},\mathbf{y},\mathbf{w})$ is zero. This implies by Lemma 6 that the linear form $f^{(1)}(\mathbf{x},\mathbf{w})$ divides f. Therefore the hyperplane H with equation $f^{(1)}(\mathbf{x},\mathbf{w})=0$ is an irreducible component of H_f containing $[\mathbf{w}]$. Note that since $[\mathbf{w}]$ is smooth on H_f it cannot belong to any other component.

The case when the irreducible factor g of f is multiple is more complicated[2]. First one establishes the identity:

$$Q_n(f) = f_1(z)^n\, Q_n(f_2) + f_2(z)^n\, Q_n(f_1) \tag{7}$$

for $f=f_1\,f_2$. This follows from:

$$\begin{aligned} &f\,(t\,\mathbf{z}-f(\mathbf{z})\,\mathbf{x})\,/\,f(\mathbf{z}) \\ &= f_1\,(t\,\mathbf{z}-f_1(\mathbf{z})\;(f_2(\mathbf{z})\mathbf{x}))\,/\,f_1(\mathbf{z})\;\cdot\; f_2\,(t\,\mathbf{z}-f_2(\mathbf{z})\;(f_1(\mathbf{z})\,\mathbf{x}))\,/\,f_2(\mathbf{z}). \end{aligned}$$

From (7) one deduces that for any two forms f_1 and f_2:

$$Q_n(f_1^k f_2) = f_1(\mathbf{z})^{n(k-1)}\;\left(k\,f_2(\mathbf{z})^n Q_n(f_1) + f_1(\mathbf{z})^n Q_n(f_2)\right).$$

Assume that $f=g^k h$, where g does not divide h, and, as before, g is the unique irreducible factor of f vanishing at $\mathbf{w}$. Then:

$$\begin{aligned} B_f &= C_{\mathsf{Apo},\mathbf{x}}(f, Q_n(f)) \\ &= g(\mathbf{z})^{n(k-1)}\;\left(k\,h(\mathbf{z})^n C_{\mathsf{Apo},\mathbf{x}}\,(f, Q_n(g)) + g(\mathbf{z})^n C_{\mathsf{Apo},\mathbf{x}}\,(f, Q_n(h))\right). \end{aligned}$$

Since B_f vanishes identically, so does

$$k\,h(\mathbf{z})^n C_{\mathsf{Apo},\mathbf{x}}\,(f, Q_n(g)) + g(\mathbf{z})^n C_{\mathsf{Apo},\mathbf{x}}\,(f, Q_n(h))\,.$$

Evaluating at $\mathbf{z}=\mathbf{w}$ we get:

$$0 = k\,h(\mathbf{w})^n C_{\mathsf{Apo},\mathbf{x}}\,(f(\mathbf{x}), Q_n(g)(\mathbf{x},\mathbf{w}))\,.$$

We have $h(\mathbf{w})\neq 0$. Therefore

$$0 = C_{\mathsf{Apo},\mathbf{x}}\,(f(\mathbf{x}), Q_n(g)(\mathbf{x},\mathbf{w}))\,.$$

We can conclude as in the first case that $g^{(1)}(\mathbf{x};\mathbf{w})$ divides f. □

[2]This difficulty has been ignored by Gordan [9] as well as in the account of the construction of Brill's covariant given in [8]. The proof given in these texts for Theorem 1 is incomplete, since it does not rule out the possibility that Brill's covariant vanish for non-totally decomposable forms whose factors are all multiples.

5. Computations in the ring $\mathcal{A}$

Let C be either Gaeta's covariant G or Brill's covariant B for forms of degree n in N variables. We have an expansion:

$$C = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma} \mathbf{x}^\alpha \mathbf{y}^\beta \mathbf{z}^\gamma.$$

The functions $c_{\alpha,\beta,\gamma}$ are homogeneous polynomials of the same degree in the coefficients a_ω of f. The system of all equations $c_{\alpha,\beta,\gamma}(f) = 0$ defines set-theoretically:

- $\mathcal{D}_n(\mathbb{K}^N)$ for $C = B$
- the union of $\mathcal{D}_n(\mathbb{K}^N)$ with the set of forms admitting a multiple factor for $C = G$.

Consider the following problem:

Problem. *Compute a linear basis for the linear span $L(C, n, N)$ of the polynomials $c_{\alpha,\beta,\gamma}$.*

Computing directly the covariant C and next extracting the coefficients, one will meet two difficulties:

- size: we are computing the huge object C, to extract from it smaller objects (its coefficients). We will recall in 5.1 how to compute sequentially the coefficients without computing the whole covariant C.
- redundancy: many coefficients $c_{\alpha,\beta,\gamma}$ are linear combinations of the others. This is partly explained by the invariance of C under SL_2 combining the variables $\mathbf{x}$ and $\mathbf{y}$. See 5.2 and 5.5.

5.1. Computing sequentially the coefficients from the source

Write for each exponent γ of the variables $\mathbf{z}$ in C:

$$C_\gamma = \sum_{\alpha,\beta} c_{\alpha,\beta,\gamma} \mathbf{x}^\alpha \mathbf{y}^\beta.$$

We have $C = \sum_\gamma C_\gamma \mathbf{z}^\gamma$. As explained in [2], the coefficients C_γ can be computed sequentially from only one of them (the *source* of C). This is the consequence of the SL_N-equivariance property of C. We recall the formulas presented in [2].

For each i between 1 and N, let $\xi_i \in \mathbb{N}^N$ be the vector whose all coordinates are zero, except the one in position i, being equal to 1:

$$\xi_1 = (1, 0, 0, \ldots, 0), \quad \xi_2 = (0, 1, 0, \ldots, 0), \ldots$$

For j between 2 and N let Δ_j be the following operator, acting on the polynomials in the coefficients a_ω of f as defined in (1):

$$\Delta_j = \sum_{\omega \text{ s.t. } \omega_1 > 0} (1 + \omega_j)\, a_{\omega + \xi_j - \xi_1} \frac{\partial}{da_\omega}.$$

Then the following relation holds for the covariant C (see [2]):

$$\Delta_j C = x_1 \frac{\partial C}{dx_j} + y_1 \frac{\partial C}{dy_j} + z_1 \frac{\partial C}{dz_j}.$$

Extracting the coefficient of $\mathbf{z}^{\gamma+\xi_j-\xi_1}$ we get:

$$\Delta_j C_{\gamma-\xi_j+\xi_1} = x_1 \frac{\partial C_{\gamma-\xi_j+\xi_1}}{dx_j} + y_1 \frac{\partial C_{\gamma-\xi_j+\xi_1}}{dy_j} + \gamma_j\, C_{\alpha,\beta,\gamma}.$$

Isolate C_γ:

$$\gamma_j\, C_\gamma = \Delta_j C_{\gamma-\xi_j+\xi_1} - x_1 \frac{\partial C_{\gamma-\xi_j+\xi_1}}{dx_j} - y_1 \frac{\partial C_{\gamma-\xi_j+\xi_1}}{dy_j}. \tag{8}$$

One uses this relation to compute the coefficients C_γ according to decreasing values of γ with respect to lexicographical ordering on $\mathbb{N}^N$. Let k be the degree of C in the variables $\mathbf{z}$. As initial values only the coefficient $C_{(k,0,0,\ldots,0)}$ is needed. We call this coefficient *the source*[3] *of the covariant* and denote it with $\mathsf{Source}(C)$. Note that $\mathsf{Source}(C)$ is the evaluation of C at $\mathbf{z} = (1, 0, 0, \ldots, 0)$. For C equal to B, T or G, it is a much smaller object than the whole covariant.

Since we are not interested in the precise value of the covariant C, but only in the space $L(C, n, N)$, we do not need to take into account the term $-x_1 \frac{\partial C_\gamma}{dx_j} - y_1 \frac{\partial C_\gamma}{dy_j}$ in (8). Indeed,

Proposition 8. *Let C be an element of $\mathcal{A}$, homogeneous of degree d in the variables $\mathbf{z}$. Then $L(C, n, N)$ is equal to the linear span of the coefficients (with respect to the variables $\mathbf{x}$ and $\mathbf{y}$) of the polynomials:*

$$\Delta_N^{\omega_N} \Delta_{N-1}^{\omega_{N-1}} \cdots \Delta_2^{\omega_2}\, \mathsf{Source}\,(C) \tag{9}$$

for all $(\omega_2, \ldots, \omega_{N-1}, \omega_N) \in \mathbb{N}^N$ such that $\omega_2 + \cdots + \omega_{N-1} + \omega_N \leq d$.

Proof. For all $k \geq 0$ let A_k be the linear span of the coefficients of C_γ (with respect to the variables $\mathbf{x}$ and $\mathbf{y}$) for all $\gamma \in \mathbb{N}^N$ such that $\gamma_1 \geq k$.

Then (8) simplifies into:

$$\gamma_j C_\gamma \equiv \Delta_j C_{\gamma-\xi_j+\xi_1} \mod A_{k+1}\mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]. \tag{10}$$

Thanks to this formula, we can prove by induction on $d-k$, starting with $d-k=0$, that the following assertion is true for all k between 0 and d:

> The vector space A_k is equal to the linear span of the coefficients of the polynomials (9) for all $(\omega_2, \ldots, \omega_{N-1}, \omega_N) \in \mathbb{N}^N$ such that $\omega_2 + \cdots + \omega_{N-1} + \omega_N \leq d - k$.

This assertion for $d - k = d$ is the assertion of the proposition. □

5.2. Removing redundancies: expression in the brackets

Let C be a homogeneous element of $\mathcal{A}$ of multidegree (d, j, k). That is, C belongs to

$$\left(S^d S^n V \otimes \left(S^j V^* \otimes S^j V^*\right)^{SL_2} \otimes S^k V^*\right)^{SL(V)}.$$

As representations of $SL(V)$ there is

$$\left(S^j V^* \otimes S^j V^*\right)^{SL_2} \cong S^{(j,j)}(V^*),$$

[3] After Hilbert [10] this term was coined by Sylvester.

the Weyl module on V^* indexed by the partition (j, j) (see [6]). This is the set of forms in the variables $\mathbf{x}$ and $\mathbf{y}$ that can be expressed as a homogeneous polynomials of degree j in the $N(N-1)/2$ *brackets* $[ij] = x_i y_j - x_j y_i$ with $i < j$. When computing an element C of $\mathcal{A}$ we should obtain such an expression, to avoid redundancy.

The brackets are not algebraically independent (except for $N \leq 3$). A linear basis for $S^{(d,d)}(V^*)$ are the products of brackets indexed by *semi-standard tableaux*. A product of brackets $[i_1 j_1][i_2 j_2] \cdots [i_d j_d]$ (where for each k we have $i_k < j_k$) is indexed by the unique $2 \times d$ array of integers

$$\begin{bmatrix} j_1 & j_2 & \cdots & j_d \\ i_1 & i_2 & \cdots & i_d \end{bmatrix}$$

whose columns are weakly increasing from left to right in lexicographic order. The array is a *semi-standard tableau* if each of its rows is weakly increasing from left to right.

An algorithm to express any SL_2-invariant polynomial as a linear combination of the products of brackets indexed by semi-standard tableaux is provided by [12].

5.3. Method

We propose the following strategy for computing a small generating set for the linear span of the coefficients of C where C is Brill's covariant B or Gaeta's covariant G:

- Compute $\mathsf{Source}(C)$ by applying the construction of Section 4 or Section 3 with $\mathbf{z}$ specialized at $(1, 0, 0, \ldots, 0)$.
- Express $\mathsf{Source}(C)$ as a linear combination of the products of brackets indexed by standard tableaux. (There might exist a better way to compute the decomposition of $\mathsf{Source}(C)$ in the brackets. For instance, for Brill's covariant, by means of the symbolic expression for the apolar covariant.).
- Apply the recurrent formulas presented in 5.1.

5.4. A toy example: ternary quadratic forms

It is well known, from the theory of quadratic forms, that a complex ternary quadratic form is totally decomposable if and only if the determinant of its matrix vanishes. We compute here Brill's covariant for the ternary quadratic forms to illustrate the methods presented in this section. We will obtain that Brill's equations are all proportional to the determinant of the matrix of the quadratic form.

The ternary quadratic form is:

$$f(\mathbf{z}) = a_{200} z_1^2 + a_{020} z_2^2 + a_{002} z_3^2 + a_{110} z_1 z_2 + a_{101} z_1 z_3 + a_{011} z_2 z_3.$$

Its polars are $f^{(0)}(\mathbf{x}, \mathbf{z}) = f(\mathbf{z})$, $f^{(2)}(\mathbf{x}, \mathbf{z}) = f(\mathbf{x})$ and

$$\begin{aligned} f^{(1)}(\mathbf{x}, \mathbf{z}) = {} & 2\, a_{200} x_1 z_1 + 2\, a_{020} x_2 z_2 + 2\, a_{002} x_3 z_3 \\ & + a_{110}(x_1 z_2 + x_2 z_1) + a_{101}(x_1 z_3 + x_3 z_1) + a_{011}(x_2 z_3 + x_3 z_2). \end{aligned}$$

We evaluate them at $\mathbf{z} = (1, 0, 0)$:

$$\begin{aligned} f^{(0)}(\mathbf{x}, (1,0,0)) &= a_{200}, \\ f^{(1)}(\mathbf{x}, (1,0,0)) &= 2\,a_{200}x_1 + a_{110}x_2 + a_{101}x_3, \\ f^{(2)}(\mathbf{x}, (1,0,0)) &= f(\mathbf{x}). \end{aligned}$$

The evaluation at $\mathbf{z} = (1, 0, 0)$ of $f(t\mathbf{z} - f(\mathbf{z})\mathbf{x})/f(\mathbf{z})$ is therefore:

$$t^2 - (2\,a_{200}x_1 + a_{110}x_2 + a_{101}x_3)\,t + a_{200}f(\mathbf{x})$$

The symmetric power sum p_2 is obtained from the elementary symmetric polynomials as $e_1^2 - 2\,e_2$. Set g for $Q_2(f)(\mathbf{x}; (1, 0, 0))$. We have:

$$\begin{aligned} g &= (2\,a_{200}x_1 + a_{110}x_2 + a_{101}x_3)^2 - 2\,a_{200}f(\mathbf{x}) \\ &= 2\,a_{200}^2\,x_1^2 + (a_{110}^2 - 2\,a_{200}\,a_{020})\,x_2^2 + (a_{101}^2 - 2\,a_{200}\,a_{002})\,x_3^2 \\ &\quad + 2\,a_{200}\,a_{110}\,x_1x_2 + 2\,a_{200}\,a_{101}\,x_1x_3 + 2\,(a_{110}a_{101} - a_{200}a_{011})\,x_2x_3 \end{aligned}$$

Therefore,

$$\mathsf{Source}(B) = C_{\mathsf{Apo},\mathbf{x}}(f, g) = 2\,f(\mathbf{y})g(\mathbf{x}) - f^{(1)}(\mathbf{x}; \mathbf{y})g^{(1)}(\mathbf{x}; \mathbf{y}) + 2\,f(\mathbf{x})g(\mathbf{y})$$

After decomposing in the brackets we obtain:

$$\mathsf{Source}(B) = -8\,[23]^2\,D$$

where $[23] = x_2y_3 - x_3y_2$ and D is the determinant of the matrix of f:

$$D = a_{200}a_{020}a_{002} + \frac{1}{4}a_{110}a_{101}a_{011} - \frac{1}{4}a_{200}a_{011}^2 - \frac{1}{4}a_{020}a_{101}^2 - \frac{1}{4}a_{002}a_{110}^2$$

After Proposition 8, Brill's equations span the same vector space as D and the coefficients with respect to $\mathbf{x}$ and $\mathbf{y}$ of the following polynomials:

$$\begin{aligned} &\Delta_2\mathsf{Source}(B), \quad \Delta_2^2\mathsf{Source}(B), \quad \Delta_3\Delta_2\mathsf{Source}(B), \\ &\Delta_3\mathsf{Source}(B), \quad \Delta_3^2\mathsf{Source}(B). \end{aligned}$$

We have:

$$\Delta_2 = a_{110}\frac{\partial}{da_{200}} + 2\,a_{020}\frac{\partial}{da_{110}} + a_{011}\frac{\partial}{da_{101}}$$

and

$$\begin{aligned} a_{110}\frac{\partial\mathsf{Source}(B)}{da_{200}} &= -8\,[23]^2\left(a_{110}a_{020}a_{002} - \frac{1}{4}a_{110}a_{011}^2\right), \\ 2\,a_{020}\frac{\partial\mathsf{Source}(B)}{da_{110}} &= -8\,[23]^2\left(\frac{1}{2}a_{020}a_{101}a_{011} - a_{002}a_{110}a_{020}\right), \\ a_{011}\frac{\partial\mathsf{Source}(B)}{da_{101}} &= -8\,[23]^2\left(\frac{1}{4}a_{110}a_{011}^2 - \frac{1}{2}a_{020}a_{101}a_{011}\right). \end{aligned}$$

Therefore $\Delta_2\mathsf{Source}(B) = 0$. Similarly we compute that $\Delta_3\mathsf{Source}(B) = 0$. This shows that Brill's equations for the ternary quadratic form are all proportional to the determinant D.

5.5. Still more redundancies

Let C be either Gaeta's covariant G or Brill's covariant B and (d, j, k) its multi-degree. After expressing the homogeneous covariant C as

$$C = \sum c_{T,\gamma} T(\mathbf{x}, \mathbf{y}) \mathbf{z}^{\gamma} \tag{11}$$

where the sum is carried over all monomials T of degree j in the brackets indexed by semi-standard tableaux and all monomials $\mathbf{z}^{\gamma}$ of degree k, we have obtained with the forms $c_{T,\gamma}$ a much smaller generating set for $L(C, n, N)$. We may expect these generators to be linearly independent. Explicit computations show the contrary: $\dim L(c, n, N)$ is still much smaller that the number of summands in (11). For instance, for $N = 4$ with $n = 3$, there are 4200 summands in (11) for $C = B$ but $\dim L(B, 3, 4) = 875$ (as reported in [5, 1]).

For ternary forms, the explanation is known: Brill's covariant can be divided by the square of

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Indeed, proving this is the object of Gordan's paper [9]. One can show that the same holds for Gaeta's covariant. Therefore, for ternary forms, there exist polynomial covariants $\widehat{C}$ (for $C = B$ or G) such that:

$$C = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 \cdot \widehat{C}$$

A symbolic form for $\widehat{B}$ is provided by [4] for $n = N = 3$ and by Gordan [9] for $N = 3$, all n.

The smaller covariant $\widehat{C}$ can be decomposed into:

$$\widehat{C} = \sum_{T,\gamma} \widehat{c}_{T,\gamma} T(\mathbf{x}, \mathbf{y}) \mathbf{z}^{\gamma}$$

where the sum is carried over all T monomials of degree $j-2$ in the brackets and all $\mathbf{z}^{\gamma}$ monomials of degree $k-2$. The forms $\widehat{c}_{T,\gamma}$ give a still smaller generating system for $L(C, n, 3)$. Explicit computation show that the cardinal of this generating set coincides with $\dim L(B, n, 3)$ for $C = B$ and $n = 4$ and $n = 5$. (For $n = 3$ the generating set has 45 elements but $\dim L(B, 3, 3) = 35$.) After this we can expect that for all $n > 3$,

$$\dim L(B, n, 3) = \binom{n}{2}^2 (n^2 - n - 1).$$

Note that the method presented in Section 5.3 is easily adapted to this simplification. It is enough to observe that for $N = 3$,

$$\mathsf{Source}(\widehat{C}) = \mathsf{Source}(C)/[23]^2.$$

It would be interesting to study how the above factorization property of B and G exhibited for ternary forms generalizes to forms of higher arity, and how to use it for more efficient computations. Brill's covariant and Gaeta's covariant belong to the ideal I generated by the maximal minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ y_1 & y_2 & \cdots & y_N \\ z_1 & z_2 & \cdots & z_N \end{bmatrix}$$

since, by construction, they vanish for all specializations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ at three coplanar vectors. We conjecture that Brill's covariant and Gaeta's covariant actually belong to the square of the ideal I.

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Symmetric Functions and Secant Spaces of Rational Normal Curves

Federico Gaeta

Abstract. The "Protean" (see [B. Sagan]) Schur functions reappear (again!) as normalized Grassmann coordinates of a subspace S_{n-1} of $\mathbb{P}_N$ n-secant of a rational normal curve $\mathcal{R}_N$, $N \geq n$. This property enables a new geometric reformulation of the theory of symmetric functions.

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The LaTeX adaptation of original Gaeta's manuscript is due to Jorge Caravantes and Laureano Gonzalez-Vega. This is an extended version of what appears in Gaeta's preprint [G2].

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1. Historical summary

We deal with an application of the algebraic Geometry of the rational normal curves $\mathcal{R}_N \subset \mathbb{P}_N$, to the Theory of symmetric functions of the n variables r_1, r_2, ..., $r_n \in \mathbb{C}$, $n \leq N$.

The so-called "Schur functions"[1] are symmetric in the n roots r_j of a polynomial F_n. They appear in many different ways as the mythological Proteus, but the relation with the elementary symmetric functions $e_1, e_2, \ldots, e_n$ was much less explicit in the literature than others such as the m, h, etc.

In this paper we prove that the coefficients of the remainders

$$R_j(\equiv x^{n-1+j} \mod F_n, \quad j = 1, 2, \ldots, N-n+1)$$

are the $n(N-n+1) = \dim \mathcal{G}(n-1;N)$ Schur functions which are certain local coordinates in the Grassmannian $\mathcal{G}$ explicitly expressed as polynomials in the e_j; they define the n-dimensional rational subvariety $\Sigma(n;N)$ of $\mathcal{G}$ locus of the n-secant spaces S_{n-1} of $\mathcal{R}_N$ spanned by the n images $(1 r_j r_j^2 \ldots r_j^N)$ of the n points r_j.

The $p^{\cdots}$ of S_{n-1} normalized by $p^{01\ldots,n-1} = 1$ i.e by the division $s = \frac{p^{\cdots}}{p^{01\ldots,n-1}}$ coincide with the Schur functions after the oldest Jacobi definition (1841), as bialternants, i.e as quotients of two alternating functions.

Some more details are summarized in the Introduction, index and in the statement and proof of the Theorem in Subsection 17.1.

[1] Because Schur studied them at the beginning of the XXth century as characters of the $GL(n,\mathbb{C})$.

2. Introduction and some prerequisites

In Part I we point out the development of the same geometrical theory sketched in the Abstract in terms of the simplest model $\mathbb{P}(S^n(\mathbf{E}_2))$[2], i.e., of the nth symmetric power $\mathbb{P}_1^{(n)}$ of the complex projective line $\mathbb{P}_1 = \mathbb{P}(\mathbf{E}_2)$. This approach relates naturally to the theory of invariants of binary Forms ([H], [K-R]) developed with the language of n-dimensional projective Geometry and the Grassmann Geometry as aimed in [R1]. In particular we have the duality between the e and the h symmetric functions appear as a consequence of Clifford's duality in $\mathbb{P}_{2n-1}$[3].

Let us summarize now with coordinates in a more down to earth way. Let $r_1, r_2, \ldots, r_n$ be n distinct roots of a monic polynomial equation $F = 0$ of degree equal to $n, F \in \mathbb{C}[x]$. Let

$$\mathbb{R} = (r_i^j) \quad i = 1, 2, \ldots, n, \qquad j = 0, 1, 2, \ldots, N \tag{2.1}$$

be the $n \times (N+1)$ matrix of normalized homogeneous coordinates of the n points P_j image of the r_j by the canonical embedding $i_N : \mathbb{P}_1 \hookrightarrow \mathbb{P}_N$ where $n \leq N$ (see Part I).

$i_N(\mathbb{P}_1)$ is the *rational normal curve* $\mathcal{R}_N$ of $\mathbb{P}_N$ parametrically represented by:

$$X_h = x^h \quad h = 0, 1, 2, \ldots, N \tag{2.2}$$

The n-minors of $\mathbb{R}$ are the Grassmann coordinates $p^{\cdots}$ of the subspace S_{n-1} of $\mathbb{P}_N$ spanned by the P_j. In particular $p^{01\ldots n-1} = V$ is a *Vandermonde* determinant $\neq 0$ dividing any $p^{\cdots}$, thus we have:

> *The Schur functions $p^{\cdots}/V$ (see* [Gi, L1, L-2, Mcd]*) reappear indeed as Grassmann coordinates of the n-secant space S_{n-1} of $\mathcal{R}_N$ normalized by $p^{01\ldots n-1} = 1$.*

Any other $p^{\cdots}$ is a $k \times k$ minor extracted from an $n \times (N-n+1)$ matrix $\mathbb{S} = (s_{ij})$ consisting of local coordinates of S_{n-1} for all the allowable values of the k's where

$$1 \leq k \leq \min(n, N-n+1)$$

the s_{ij} are Schur functions of the *hook type*, in particular the first "h"-row describes the intersection of S_{n-1} with a well-defined coordinate space.

> *This paper aims for a reconstruction of a geometric theory of symmetric functions by means of the above interpretation.*

At the very beginning the classical theory of symmetric functions dealt with n points variable in the complex affine line $\mathbb{A}_1$ but actually $\mathbb{A}_1$ was taken for granted, almost "forgotten" and it was disguised in various abstract ways in the

[2]We assumed as well-known the properties of the symmetric algebras $S(\mathbf{E}_2) = \oplus_0^\infty S^n(\mathbf{E}_2)$, $S(\mathbf{E}_2^\vee)$ isomorphic with $\mathbb{C}[\mathbf{u}_0, \mathbf{u}_1]$ (resp. with $\mathbb{C}[x_0, x_1]$) where $(\mathbf{u}_0, \mathbf{u}_1)$ and (x_0, x_1) are arbitrarily fixed dual basis of $\mathbf{E}_2, \mathbf{E}_2^\vee$, see [N].

[3]Our starting point as a non expert in symmetric functions is the vague statement that the symmetric functions of the roots of F are functions of the coefficients. All the references to the modern formal theory of symmetric functions use the [Mcd] notations, e.g., the e, h, s. However for us they are again true functions $\mathbb{A}_1^{(n)} \mapsto \mathbb{C}$, e.g., $(r_1, r_2, \ldots r_n) \mapsto e(r_1, r_2, \ldots, r_n)$.

new formal theory, see the standard reference [Mcd], but it is clear that $\mathbb{A}_1$ might be replaced by any other smooth algebraic curve bijectively (better: birationally and biregularly) related to $\mathbb{A}_1$ such as the mentioned curve $\mathcal{R}_N$ defined by (2.2) in $\mathbb{A}_n$ in a suitably chosen coordinate system by the injective map $i_n : \mathbb{A}_1 \to \mathbb{A}_n$ defined by

$$i_n : x \mapsto (x, x^2, \dots, x^n) \quad \forall x \in \mathbb{C}. \tag{2.3}$$

(see [BE, E, G1]).

Any hyperplane

$$a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = 0$$

of $\mathbb{A}_n$ meets $\mathcal{R}_n$ in a finite number of points $(r_j, r_j^2, \dots, r_j^n)$ images of the roots r_j of the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \tag{2.4}$$

with the same coefficients than F. If $a_n \neq 0$ then this polynomial has degree equal to n. We make a few very close algebraic-geometric statements: the equation in (2.4) has n roots provided we count each one with its well-defined multiplicity, i.e., we need to take into account the *intersection multiplicities* (see [F2], [F3]). We attach the so called *divisor*

$$\mathbf{D}_n = m_1 R_1 + m_2 R_2 + \cdots + m_h R_h$$

of degree n to the algebraic equation in (2.4) of degree n with $h(\leq n)$ distinct roots R_j (with affine coordinates r_j) and multiplicities m_j where

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_n \prod_{j=1}^{h} (x - r_j)^{m_j},$$

with $n = m_1 + m_2 + \cdots + m_h$.

Informally we state: a divisor of $\mathbf{D}_n$ of degree n is an unordered set of points in the projective line $\mathbb{P}_1 = \mathbb{A}_1 \cup \mathbf{I}_\infty$ counting each one with a positive multiplicity. The "number n of points of $\mathbf{D}_n$" equals the sum of the multiplicities.

Definition 2.1. We shall fix throughout the paper the notation F (or F_n) for a monic ($\Leftrightarrow a_n = 1$) polynomial

$$F = \prod_{j=1}^{n} (x - r_j) = x^n - e_1 x^{n-2} + \cdots + (-1)^n e_n \tag{2.5}$$

If the discriminant of F is $\neq 0$ there are n simple roots $\Leftrightarrow$ all the m are equal to 1, and then $h = n$.

We can remove the restriction $a_n \neq 0$ by enlarging – in the usual way – both affine spaces to the corresponding projective ones: $\mathbb{A}_1 \subset \mathbb{P}_1, \mathbb{A}_n \subset \mathbb{P}_n$ introducing *the points at infinity*. If the leading coefficient a_n vanishes one of the n roots "goes to ∞". Then formally degree F becomes less than n because with the absolute coordinates we cannot handle properly the point at infinity $\mathbf{I}_\infty$. Let us introduce

the homogeneous coordinates (x_0, x_1) in $\mathbb{P}_1$ fixed also in the sequel: $\mathbb{P}_1 = \mathbb{A}_1 \cup \mathbf{I}_\infty$. Let us call again $i_n : \mathbb{P}_1 \to \mathbb{P}_n$ the embedding map:

$$(x_0, x_1) \mapsto (x_0^n, x_0^{n-1}x_1, \dots x_1^n) \quad \forall (x_0, x_1) \in \mathbb{P}_1 \tag{2.6}$$

(see [BE]), natural extension of (2.3) expressed now in the corresponding shown homogeneous coordinates $(X_0, X_1, \dots, X_n)$ in $\mathbb{P}_1$ (resp. in $\mathbb{P}_n$) where $x = x_1/x_0$. The image of $\mathbf{I}_\infty$ in $\mathcal{R}_n$ is the unique point at infinity $n\mathbf{I}_\infty$ of $\mathcal{R}_n$ with homogeneous coordinates $(0, 0, \dots, 0, 1)$. The hyperplane at infinity: $X_0 = 0$ of $\mathbb{P}_n$ osculates $\mathcal{R}_n$ in $n\mathbf{I}_\infty$ (see §7).

In this enlarged projective framework we are allowed to consider also $\mathbf{I}_\infty$ as a possible root after replacing the unique F by a binary form f.

Definition 2.2. We shall fix throughout the paper the notation f (or f_n) for a binary form $\neq 0$ of degree n (well defined up to a zero constant factor): $f \cong \lambda f$, for all $\lambda \in \mathbb{C}^\times$

$$f = f_n = a_0x_0^n + a_1x_0^{n-1}x_1 + a_2x_0^{n-2}x_1 + \dots + a_nx_1^n. \tag{2.7}$$

Again: solving the homogeneous equation $f = 0$ defining $\mathbf{D}_n$ or intersecting $\mathcal{R}_n$ with the hyperplane $a_0X_0 + a_1X_1 + a_2X + \dots + a_nX_n = 0$ with the same a_j's looks as a matter of taste ....

If $a_n = 0$ we can write f_n in the form: $f_n = x_0^k F_{n-k}$ where x_0 does not divide F_{n-k}. $k \geq 1$ is the multiplicity of $\mathbf{I}_\infty$ in the projective divisor $\mathbf{D}_n = k\mathbf{I} + \mathbf{A}_{n-k}$ defined by $f = 0$. Then F_{n-k} will be always assumed to be monic.

Definition 2.3. A divisor $\mathbf{D}_n$ not containing $\mathbf{I}_\infty$, i.e., containing only finite roots as before will be called **an affine divisor** $\mathbf{A}_n (\Leftrightarrow \mathbf{A}_n \in \mathbb{A}_1^{(n)})$.

The projective divisors $\mathbf{D}_n$ ($\Leftrightarrow$ just divisors for short) are naturally classified attending to the multiplicity $\gamma(\mathbf{I}_\infty) = k$ in $\mathbf{D}_n$ as before:

$$\mathbf{D}_n = k\mathbf{I}_\infty + \mathbf{A}_{n-k}, \ k = 0, 1, 2, \dots, n.$$

Examples. Let us consider the initial cases $N = 2, 3, \dots$. No "chord" makes sense for a couple of distinct points A, B in the line, but in the parabola $\mathcal{R}_2$ we can easily see the chord joining A and B – even if the two points are identical. Then we write $2A$ and we visualize the chord as the tangent line to $\mathcal{R}_2$ at A. It is easy to see the chord of $A + B$ or $2A$ in the cubic $\mathcal{R}_3$.

We have similar troubles for divisors $A+B+C$ if we try to see "trisecants" in $\mathcal{R}_1$ or $\mathcal{R}_2$, but there is no difficulty in recognizing that any divisor $\mathbf{D}_3$ is a section of the twisted cubic $\mathcal{R}_3$ by a plane $u_0 + u_1X_1 + u_2X_2 + u_3X_3 = 0 \Leftrightarrow$ in solving the equation $u_0 + u_2x + u_2x^2 + u_3x^3 = 0$ obtained from (2.4). Actually there is no difficulty in the construction of the equation $(x - a)(x - b)(x - c) = f(x)$ with $u_3 = 1$ corresponding to the divisor $A+B+C$ using the corresponding coordinates a, b, c independently of the three cases $A + B + C, 2A + B, 3C$ corresponding to three distinct points or to the plane joining the tangent line at $A(= (a, a^2, a^3))$ and the point $B = (b, b^2, b^3) \neq A$ or considering the osculating plane to $\mathcal{R}_3$ at C if we keep the same notations $A, B, C, \dots$ for points in the line or in $\mathcal{R}_3$.

The typical procedure if we fix $n = \deg \mathbf{D}$ is to introduce some $\mathcal{R}_N$ with $N > n$. $\mathcal{R}_N$ is an irreducible algebraic rational curve of degree N – well defined up to a collineation of the ambient space $\mathbb{P}_N$; it was introduced because of its *normality* ($\Leftrightarrow$ any positive divisor $\mathbf{D}_N$ in $\mathbb{P}_1$ becomes a hyperplane section of $\mathcal{R}_N$ [BE, G1]), it is the natural hyperspace copy of $\mathbb{P}_1$ in $\mathbb{P}_N$ *not contained in a hyperplane.* $\mathcal{R}_N$ is invariant by a transitive collineation group in $\mathbb{P}_N$ isomorphic with $\mathbb{P}GL(\mathbf{E}_2) \Rightarrow \mathcal{R}_N$ is smooth!

Any $\mathcal{R}_N$ (for $N \geq n$) is well adapted to the study of symmetric functions of a given number n of variable points in the line. We realized since §0 that such interpretation arises in the classical formulas on symmetric functions via Jacobi's oldest definition of the Schur functions! (see [J]).

The study of the manifold $\mathbb{P}_m^{(n)} = \mathbb{P}_m^n/\mathcal{S}_n$ (see [G1]) representing the informally called unordered n-tuples of points in the complex space $\mathbb{P}_m = \mathbb{P}(\mathbf{E}_{m+1})$ relates closely to its total branch locus: the Veronese variety $\mathcal{V}_m^{(n)}$ of $\mathbb{P}_m$. It would be very handy to read a paper on the simplest case $m = 1$, then $\mathcal{V}_1^{(n)}$ is a *rational normal curve* $\mathcal{R}_n$ in $\mathbb{P}_n$, but we did not find elsewhere this needed geometric approach (see the forthcoming [G3]).

This paper intends to fill this gap in the literature.

The Algebraic Geometry needed here is the most elementary one (just for algebraic curves of genus equal to zero (birationally equivalent to $\mathbb{P}_1$)). It appears very natural looking at the geometrical meaning of the classical formulas, for symmetric functions although many of them are much older than Grassmann and the Geometry, of algebraic curves, e.g., Newton is frequently quoted!

Looking at the possible extensions to $\mathbb{P}_m^{(n)}$ in September 1998 we must take into account this algebraic geometric approach; the Veronese variety $\mathcal{V}_m^{(n)}$ in $\mathbb{P}(S^n(\mathbf{E}_{m+1}))$ has the same intrinsic definition as $\mathcal{R}_n$ and its parametric representation by means of homogeneous monomials of degree n in $(m+1)$-ary Forms suggests a close relation with generating functions much harder to handle that for formal series in one variable. It seems worthwhile to look for a serious approach transforming heuristic Remarks in Theorems.

The algebraic-geometric structure of the line is very simple: just the linear equivalence of divisors in order to define the normal embeddings. Classical algebraic geometry uses frequently different projective models of a given algebraic variety in a suitably chosen affine or projective space, depending on the problem.

We hope that the reader – by a routine inspection of the standard formulas for symmetric functions – found with us – that all such embeddings of the line in $\mathbb{P}_N$, $N = 1, 2, \ldots$ were actually used implicitly: Namely if we want to study symmetric functions of n points all the $\mathcal{R}_N$ of $\mathbb{P}_1$ for $N \geq n$ were used via generating functions when all the powers

$$1, x, x^2, \ldots, x^n, \ldots, x^N$$

needed in the coordinate definition of the $\mathcal{R}_N$ appear constantly as a *Leitmotiv.* Without an explicit consideration of such embeddings of $\mathbb{P}_1$ the whole procedure

looks as a big "*Formulaire raisonné*". *With our geometrically biased approach the Schur functions are not more subtle than other symmetric ones and its protean character can be explained via ubiquity: they appear for every dimension* $N \geq n$.

A hyperplane section $\mathbf{H}_n$ of a rational curve is linearly equivalent to any other positive divisor $\mathbf{D}_n$ of the same degree which is not necessarily a hyperplane section, e.g., this is the case for a plane cubic with a node or for the non singular rational quartic in $\mathbb{P}_3$ projection of an $\mathcal{R}_4$ in $\mathbb{P}_4$ from an outside point. *The normality is equivalent to the completeness of the linear system of hyperplane sections* $\Leftrightarrow$ Any positive divisor linearly equivalent to a hyperplane section of degree equal to the degree of Γ is a hyperplane section.

A fixed map γ transforms naturally the line in $\mathcal{R}_n$ well defined up to the chosen γ. A suitable γ enables to represent parametrically the coordinates of $\mathcal{R}_n$ in the form (11.5), see §5.

A more detailed synthesis of our results is offered to a knowledgeable reader in the Theorem in §17.1 using previous definitions, prerequisites, lemmas and cross references. There we also point out some possible new approaches and formulas which we were not able to find in the literature.

Part I. An Intrinsic Coordinate Free Approach of $\mathbb{P}_1^n$

3. First definitions

A point of $\mathbb{P}_1^{(n)} = \mathbb{P}_1^n/S_n$, i.e., an unordered n-tuple of points in $\mathbb{P}_1$ is formally identified with a *positive divisor* of $\mathbb{P}_1$.

Definition 3.1. Formally a divisor $\mathbf{D}$ of an irreducible projective algebraic complex curve Γ is an element

$$\mathbf{D} = \sum_{P\in\Gamma} m_p P$$

of the free module $\mathbb{Z}(\Gamma)$ over the set of points of Γ. The well-defined integer

$$\sum m_P = n \gtreqless 0$$

is called the degree of $\mathbf{D} = \mathbf{D}_n$.

A $\mathbf{D}_n \neq 0$ is called *positive* if and only if $m_P \geq 0$ for all $P \in \Gamma$. We use here just the positive ones and we omitted this adjective, but in the definition of linear equivalence of divisors (denoted by $\equiv$) we need to attach a divisor $\operatorname{div} R$ to any non zero rational function R where $\operatorname{div} R = \mathbf{Z} - \mathbf{P}$ (*divisor of zeros of R minus divisor of poles*). Namely we define: $\mathbf{D}_1 \equiv \mathbf{D}_2$ if and only if $\mathbf{D}_1 - \mathbf{D}_2 = \operatorname{div} R$ for some R, for example any two sections of a $\Gamma \subset \mathbb{P}_n$ by hyperplanes $L_1 = 0$. $L_2 = 0$ are linearly equivalent because L_1/L_2 is an R (see [F1, F2]).

The set of all the positive divisors in a linear equivalence class is called a *complete linear system*. It has a natural structure of projective space.

In $\mathbb{P}_1$ ($\Leftrightarrow$ in any birationally curve equivalent to $\mathbb{P}_1$ such as the $\mathcal{R}_n$ for any $n = 1, 2, \ldots$ which is the *only case needed here*) the linear equivalence is trivial ($\Leftrightarrow$ any two divisors of the same degree are linearly equivalent).

All the complete linear systems in $\mathbb{P}_1$ have the type $\{\mathbf{D}_n > 0; \mathbf{D}_n \equiv nP\}$ for any $n = 1, 2, \ldots$ and $P \in \mathbb{P}_1$. It will be denoted by $|\mathbf{D}_n|$ for short. The dimension of $|\mathbf{D}_n|$ equals n. We can check it immediately because any $\mathbf{D}_n$ is defined by some binary form f defined up to a non zero constant factor $\Rightarrow$ the coefficients of f in a suitable prefixed ordering can be taken as homogeneous coordinates ($\Leftrightarrow$ coefficients) of a hyperplane in $\mathbb{P}_n = \mathbb{P}(\mathbf{E}_{n+1})$.

For instance in a conic Γ , A + B + 2C is a positive divisor if A, B, C are three different points, 2A is the point A counted twice. The *intersection divisor* $\Gamma.H$ of a given Γ with a hypersurface $H(\not\supset \Gamma)$ contains the set theoretic intersection $\Gamma \cap H$ but counting each point P with its multiplicity $m_P = i(P, \Gamma, H)$. Formally, $m_P(\geq 0)$ is the intersection multiplicity. In the previous conic m_P = A + B is the intersection $\Gamma.r$ with a secant line, 2A is the intersection cycle of Γ with the tangent line to Γ at A.

The starting point is the *divisor* $\mathbf{D}_n = \sum m_P P > 0$ *of a homogeneous equation* $f_n = 0$ where $m_P \geq 0$ denotes the multiplicity of the point P as a root ($m_P = 0$ iff P is not a root and $\sum m_P = n$). As usual we shall write only the P where $m_P > 0$.

4. The nth symmetric power of the projective lines $\mathbb{P}_1$ and $\mathbb{P}_1^\vee$

We shall identify $\mathbb{P}_1^{(n)}$ and $\mathbb{P}_1^{\vee(n)}$ with the projective spaces $\mathbb{P}(S^n(\mathbf{E}_2))$ and $\mathbb{P}(S^n(\mathbf{E}_2^\vee)) = (\mathbb{P}(S^n(\mathbf{E}_2)))^\vee$ arising from the shown symmetric powers of the main vector space $\mathbf{E}_2$ and its dual. Any non zero element $\mathbf{f} \in S^n(\mathbf{E}_2)$ has the form $\mathbf{f} = \mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_\mathbf{n}$, where $\mathbf{r}_i \in \mathbf{E}_2 - \mathbf{0}$, $i = 1, 2, \dots, n$ (see [N]). Since this prime factor decomposition is essentially unique ($\Leftrightarrow$, i.e., up to the ordering and non-zero constant factors) the canonical projection $(\mathbf{f}) = \mathbb{P}(\mathbf{f})$ of a non zero $\mathbf{f}$ is a point of $\mathbb{P}(S^n(\mathbf{E}_2))$ representing a divisor $\mathbf{D}_n$ of degree n of $\mathbb{P}_1$, and conversely, in particular for $n = 1$ we come back to the points of $\mathbb{P}_1$ since we denote by $A = (\mathbf{a})(= (c\mathbf{a})$, any $c \in \mathbb{C}^\times)$, short for $\mathbb{P}(\mathbf{a}) \in \mathbb{P}_1$, the canonical projection of the non zero representative vector $\mathbf{a} \in \mathbf{E}_2$. Accordingly:

the nth symmetric power $\mathbb{P}_1^{(n)} = \mathbb{P}_1^n/\mathcal{S}_n$ (see [G2]) *is naturally identified with* $\mathbb{P}(S^n(\mathbf{E}_2))$ *by the birational and biregular map*

$$i_n : \mathbb{P}_1^{(n)} \leftrightarrow \mathbb{P}(S^n(\mathbf{E}_2))$$

intrinsically and canonically well defined by

$$A_1 + A_2 + \cdots + A_n \mapsto (\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n) \in \mathbb{P}(S^n(\mathbf{E}_2)) \tag{4.1}$$

and we assume $A_j = (\mathbf{a}_j)$, $j = 1, 2, \dots, n$.

Frequently we shall emphasize non proportional factors as a natural exponential notation $(\mathbf{r}_1^{m_1}\mathbf{r}_2^{m_2} \dots \mathbf{r}_h^{m_h})$ for a divisor:

$$\mathbf{D}_n = m_1R_1 + m_2R_2 + \cdots + m_hR_h = (\mathbf{r}_1^{m_1}\mathbf{r}_2^{m_2} \dots \mathbf{r}_h^{m_h}). \tag{4.2}$$

If $n > 1$ we obtain the expression:

$$\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n = \prod_{j=1}^{n} (a_{j0}\mathbf{u}_0 + a_{j_1}\mathbf{u}_1)^n = \sum_{j=0}^{n} \hat{e}_j \mathbf{u}_1^j \mathbf{u}_0^{n-j} \tag{4.3}$$

in terms of $\mathbf{u}$, where the $\hat{e}_j$ naturally defined by

$$\hat{e}_j(\dots, (a_{r0}, a_{r_1}), \dots) = a_{11}a_{21} \dots a_{j_1}a_{j+1,0} \dots a_{n0} + \cdots \tag{4.4}$$

are the homogeneous versions of the elementary symmetric functions of the homogeneous coordinates of the n points in $\mathbb{P}_n$ defined by symmetrization of the first subindices $1, 2, \dots, n$. The (4.4) simplify to the standard ones if none of the n distinct points is $\mathbf{I}_\infty$. Then (4.3) leads to the following coordinate expression of an affine divisor $\mathbf{A}_n = (\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_n)$ corresponding to the same roots $\mathbf{r}_j = \mathbf{u}_0 + r_j\mathbf{u}_i$, $j = 1, 2, \dots$ in terms of the basis $(\mathbf{u}_0, \mathbf{u}_1)$ defining our projective frame of reference $\mathbf{O} = (\mathbf{u}_0), \mathbf{I} = (\mathbf{u}_1), \mathbf{U} = (\mathbf{u}_0 + \mathbf{u}_1)$; the monic polynomial is still F.

Let

$$\mathbf{f} = \mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_n = (\mathbf{u}_0 + r_1\mathbf{u}_1) \dots (\mathbf{u}_0 + r_2\mathbf{u}_1) = \sum_{j=0}^{n} e_j \mathbf{u}_1^j \mathbf{u}_0^{n-j}$$

be its coordinate expression; dividing by $\mathbf{u}_0^n$ and writing $\mathbf{u} = \mathbf{u}_1/\mathbf{u}_0 \in \mathbb{C}(\mathbf{u}_0, \mathbf{u}_1)$ we have the polynomial of constant term equal to 1:

$$\mathbf{f}/\mathbf{u}_0^n = 1 + e_1\mathbf{u} + e_2\mathbf{u}^2 + \cdots + e_n\mathbf{u}^n \tag{4.5}$$

in terms of the values of the elementary symmetric functions $e_j(r_1, r_2, \ldots, r_n)$ attached to the affine divisor $\mathbf{A}_n \in \mathbb{A}^{(n)_1}$.

Definition 4.1. The polynomial $g = \mathbf{f}/\mathbf{u}_0^n = 1 + e_1\mathbf{u} + e_2\mathbf{u}^2 + \cdots + e_n\mathbf{u}^n$ of formal degree n will be called the **generating function of $\mathbf{A}_n$**.

Remark 4.2. The actual degree of g might be less than n if e_n vanishes ($\Leftrightarrow$ if at least one of the r_i becomes $= 0$). We shall choose the point $O \neq r_i$ for $i = 1, 2, \ldots, n$ in the sequel, thus $e_n \neq 0$. We can regard each e_j as a polynomial symmetric function of $r_1, r_2, \ldots, r_n$).

Remark 4.3. F and g are related by:

$$F(x) = x^n g(-x^{-1}) \qquad g(\mathbf{u}) = \mathbf{u}^n F(-\mathbf{u}^{-1}) \tag{4.6}$$

expressing that the roots of g are $p_j = -r_j^{-1}$ $(j = 1, 2, \ldots, n)$. We shall find an interpretation of this property in terms of the "dual line."

5. Orbits, intrinsic definition of the rational normal curves $\mathcal{R}_n$

The complete projective group $\mathbb{P}(GL(\mathbb{C}^{n+1}))$ of $\mathbb{P}_n(\approx \mathbb{P}(S^n(\mathbf{E}_2))$ does not preserve the decomposition in linear factors $\Rightarrow$. We shall consider the group $\mathbb{P}(S^n(GL(\mathbf{E}_2)))$ as the automorphism group of $\mathbb{P}(S^n(\mathbf{E}_2))$ in the sense of the Erlangen program when we regard $\mathbb{P}(S^n(\mathbf{E}_2))$ as the locus of all positive n-divisors in the projective line. This is the natural geometry of binary forms of degree n appearing in the study of invariants (see [H], [K-R]).

In this Geometry the space $\mathbb{P}(S^n(\mathbf{E}_2))$ is not homogeneous anymore. The first natural task is to find the orbits of our group in the action

$$PGL(\mathbf{E}_2)) \times \mathbb{P}(S^n(\mathbf{E}_2)) \to \mathbb{P}(S^n(\mathbf{E}_2)).$$

There is a general family consisting of divisors with distinct points: $\mathrm{A}_1 + \mathrm{A}_2 + \cdots + \mathrm{A}_\mathrm{n}$ where $\mathrm{A}_\mathrm{i} \neq \mathrm{A}_\mathrm{j}$ for $i \neq j$, which is the main one in the theory of symmetric functions, but in general *the multiplicities are invariant*

$$\sum m_P P \mapsto \sum m_P \sigma(P)$$

for any projectivity σ.

A very important (closed) orbit is the locus $\{nA | A \in \mathbb{P}_2\}$ of single points of the line counted n-times.

Definition 5.1. In the previous identifications of the n-divisors in the projective line $\mathbb{P}_1 = \mathbb{P}(\mathbf{E}_2)$ with points in $\mathbb{P}(S^n(\mathbf{E}_2))$ appears naturally the total branch locus $\{nA | A \in \mathbb{P}_1\}$ well defined also by:

$$\mathcal{R}_n =^{\mathrm{def}} \{(\mathbf{a}^n) | (\mathbf{a}) \in \mathbb{P}_1\} \subset \mathbb{P}(S^n(\mathbf{E}_2)) \tag{5.1}$$

called the **rational normal curve** because it is indeed a curve of degree n in n-space which cannot be the projection from an outside point to $\mathbb{P}_n$ of another Γ strictly belonging to $\mathbb{P}_{n+1}$ of the same degree. Moreover: There is a natural bijective rational map:

$$A \mapsto nA = (\mathbf{a}^n) \quad \forall A = (\mathbf{a}) \in \mathbb{P}_1. \tag{5.2}$$

We shall check that $\mathcal{R}_n$ can be represented by

$$X_0 : X_1 : \cdots : X_n = x_0^n : x_0^{n-1}x_1 : x_0^{n-2}x_1^2 : \cdots : x_1^n \tag{5.3}$$

in a suitable projective frame in $\mathbb{P}_n$ or by (2.2) for normalized coordinates if $X_0 = 1$. Indeed the binomial formula can be written as follows:

$$\mathbf{x}^n = \sum x_0^i x_1^{n-i} \binom{n}{i} \mathbf{u}^i \mathbf{u}^{n-i}. \tag{5.4}$$

Definition 5.2. Let

$$S^n(\mathbf{u}_0, \mathbf{u}_1) = (\mathbf{u}_0^n, n\mathbf{u}_0^{n-1}\mathbf{u}_1, \ldots, \binom{n}{i}\mathbf{u}_0^i\mathbf{u}_1^{n-i}, \ldots, \mathbf{u}_1^n)$$

be a basis of $S^n(\mathbf{E}_2)$ functorially attached to $(\mathbf{u}_0, \mathbf{u}_1)$. Let $\mathbb{P}(S^n(\mathbf{u}_0, \mathbf{u}_1))$ be the corresponding projective frame in $\mathbb{P}(S^n(\mathbf{E}_2))$ consisting of the following $n+1$ fundamental points: $\mathbf{O} = (\mathbf{u}_0^n), \mathbf{1} = (\mathbf{u}_0^{n-1}\mathbf{u}_1), \ldots, \mathbf{i} = (\mathbf{u}_0^{n-i}\mathbf{u}_1^i), \ldots, \mathbf{n} = (\mathbf{u}_1^n)$ plus the unit point $n\mathbf{U} = (\mathbf{u}_0 + \mathbf{u}_1)^n$.

In other words: $\mathbb{P}(S^n(\mathbf{u}_0, \mathbf{u}_1))$ contains the three images $n\mathbf{O}, n\mathbf{I}, n\mathbf{U}$ of $\mathbf{O}, \mathbf{I}, \mathbf{U}$. We shall see that the remaining fundamental points are intersections of osculating spaces $\mathbb{T}_r(\mathbf{u}_0^i)$, $\mathbb{T}_{n-r}(\mathbf{u}_1^{n-i})$ of $\mathcal{R}_n$ at $(\mathbf{u}_0^n)$, $(\mathbf{u}_1^n)$ of complementary dimensions.

The formula (5.4) tells us that *in the previous coordinate system the curve defined by* (5.1) *is represented parametrically by* (5.3).

6. Duality

The symmetry between the two pairs of homogeneous variables (a_0, a_1) and $)\alpha_0, \alpha_1($ in the "hyperplane equation"

$$\alpha_0 a_0 + \alpha_1 a_1 = 0$$

in $\mathcal{R}_1$) leads naturally to introducing the symmetric n^{th} powers $S^n(\mathbf{E}_2)$, $S^n(\mathbf{E}_2^\vee)$, of both dual vector spaces $\mathbf{E}_2$ and $\mathbf{E}_2^\vee$ (see [N]) as well as the duality functorially induced by $(u, x) \mapsto \langle u, x\rangle \in \mathbb{C}$:

$$(f, \phi) \mapsto \langle f, \phi\rangle \qquad f \in S^n(\mathbf{E}_\mathbf{2}^\vee), \phi_n \in S^n(\mathbf{E}_\mathbf{2}^\vee), \tag{6.1}$$

i.e., we assume in the sequel the identification

$$S^n(\mathbf{E}_2^\vee) = (S^n(\mathbf{E}_2))^\vee \tag{6.2}$$

Accordingly $\mathbb{P}(S^n(\mathbf{E}_2^\vee))$ is identified with the space of hyperplanes ($\Leftrightarrow$ hyperplanar divisors $\mathcal{A}_1+\mathcal{A}_2+\cdots+\mathcal{A}_n=$) $\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_n($ where $\mathbf{a}_j\in\mathbf{E}_2^\vee-\{\mathbf{0}\}$). The natural problem arises of comparing both representatives:

$$A_1+A_2+\cdots+A_n\leftrightarrow\mathcal{A}_1+\mathcal{A}_2+\cdots+\mathcal{A}_n \tag{6.3}$$

To begin with let us establish intrinsically the canonical bijection $\mathbb{P}_1\leftrightarrow\mathbb{P}_1^\vee$:

If $\mathbf{a}\neq\mathbf{0}$ is a vector of $\mathbf{E}_2$ the orthogonal $\{\,\mathbf{a}|\langle\,\mathbf{a},\mathbf{a}\rangle=0\}$ is a one dimensional space $\mathcal{A}=)\mathbf{a}(=)\lambda\mathbf{a}($ and $\mathcal{A}=\mathcal{A}'$ iff $\mathbf{a}'=\lambda\mathbf{a}$ thus $(\mathbf{a})\leftrightarrow)\mathbf{a}($ is a bijection. In other words we have:

> *The dual line* $\mathbb{P}_1^\vee=\mathbb{P}(\mathbf{E}_2^\vee)$. Any point $A=(\mathbf{a})\in\mathbb{P}(\mathbf{E}_2)=\mathbb{P}_1$ can be identified with a "hyperplane" $\tilde{\mathcal{A}}$ of $\mathbb{P}_1$ denoted also by $)\mathbf{a}($ (where $\mathbf{a}$ belongs to $\mathbf{E}_2^\vee-0$) $\Leftrightarrow$ with a point of the dual line $\mathbb{P}_1^\vee=\mathbb{P}(\mathbf{E}_2^\vee)$, see [N]; we need the well-known canonical duality between $S^n(\mathbf{E}_2^\vee)$ and $S^n(\mathbf{E}_2)$ induced by the main $\langle,\rangle$ (see footnote (1)).

Let $\alpha_0a_0+\alpha_1a_1=0$ ($\Leftrightarrow\ \langle\mathbf{a},\mathbf{a}\rangle=0$) be the trivial equation of $(\mathbf{a})$ as a hyperplane $)\mathbf{a}($ defined with both kind of homogeneous coordinates, $(a_0,a_1),(\alpha_0,\alpha_1)$.

Definition 6.1. The following map $\Pi_1:\mathbb{P}(\mathbf{E}_2)\leftrightarrow\mathbb{P}(\mathbf{E}_2^\vee)$ characterized by

$$\alpha_0=a_1\qquad\qquad\alpha_1=-a_0 \tag{6.4}$$

is called the *trivial polarity* of the projective line.

Π_1 represents the identity, but changing the role of point $(\mathbf{a})$ and hyperplane $)\mathbf{a}($ for every point.

6.1. A recall on polarities

See [BE] and [H-P] for further details. Let $\gamma:\mathbf{E}\to\mathbf{F}$ and ${}^t\gamma:\mathbf{F}^\vee\to\mathbf{E}^\vee$ be a linear bijective map between two vector spaces and its transposed. Their projections $K=\mathbb{P}(\gamma),:\mathbb{P}(\mathbf{E})\to\mathbb{P}(\mathbf{F})$ and ${}^tK=\mathbb{P}({}^t\gamma):\mathbb{P}(\mathbf{F}^\vee)\to\mathbb{P}(\mathbf{E}^\vee)$ are well defined *collineations*.

In the particular case $\mathbf{F}=\mathbf{E}^\vee=\mathrm{Hom}(\mathbf{E},\mathbb{C})$, $K=\mathbb{P}(\gamma),:\mathbb{P}(\mathbf{E})\to\mathbb{P}(\mathbf{E}^\vee)$ is frequently called a *correlation*. K maps points in hyperplanes. Then we have a *transposed correlation* ${}^tK=\mathbb{P}({}^t\gamma):\mathbb{P}(\mathbf{E})\to\mathbb{P}(\mathbf{E}^\vee)$ because of the canonical identification $(\mathbf{E}^\vee)^\vee=\mathbf{E}$. It is particularly important in classical Projective Geometry the case ${}^tK=K$ which implies $\gamma=c^t\gamma\Rightarrow c^2\gamma=\gamma$ for some $c\in\mathbb{C}\Rightarrow c=\pm1$. Conversely either one of these conditions for c implies that $\mathbb{P}(\gamma)$ is identical with its transposed. The condition $\gamma=\pm{}^t\gamma$ is satisfied iff the matrix $C=(c_{ij})$ representing γ in any coordinate system is *symmetric* or *antisymmetric* : $C=\pm{}^tC$. There is an important distinction of both previous alternatives:

There are non singular symmetric $(n+1)\times(n+1)$ matrices for every n, e.g., $C=\mathbf{1}_n$. On the other hand if n is even every $(n+1)\times(n+1)$ antisymmetric matrix has zero determinant. If $n+1=2\mu$ there are always antisymmetric non singular matrices.

Another convenient equivalent way of discussing both alternatives arises in terms of the *evaluation bilinear map* $\mathbf{C} : \mathbf{E}^\vee \times \mathbf{E} \to \mathbb{C}$ attached to γ defined by:

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \langle \gamma \mathbf{x}, \mathbf{y} \rangle,$$

namely: $\mathbf{C}(\mathbf{x}, \mathbf{y}) = \pm \mathbf{C}(\mathbf{y}, \mathbf{x})$ in the symmetric (resp. in the antisymmetric) case.

Definition 6.2. A non singular correlation $K = \mathbb{P}(\gamma)$ such that $\mathbb{P}({}^t\gamma) = \mathbb{P}(\gamma)$ is called a **polarity**. The condition $\mathbf{C}(\mathbf{x}, \mathbf{y}) = 0$ ($\Leftrightarrow \mathbf{C}(\lambda \mathbf{x}, \mu \mathbf{y}) = 0$) is meaningful for couples of points $(\mathbf{x})$, $(\mathbf{y})$ in $\mathbb{P}_n$. We say that $(\mathbf{x})$, $(\mathbf{y})$ are conjugate with respect to $\mathbb{P}(\gamma)$.

In the symmetric case the locus $Q(\mathbf{x}) = \mathbf{C}(\mathbf{x}, \mathbf{x}) = 0$ is a quadric of maximal rank and $\mathbb{P}(\gamma)$ is the polarity with respect to Q as for conics and quadrics in elementary teaching. In the case $c_{ij} = -c_{ij}$ the condition $\mathbf{C}(\mathbf{x}, \mathbf{x}) = 0$ is satisfied identically $\Leftrightarrow$ every point belongs to its polar hyperplane. Such polarities are called *Nullpolarities*: Π_1 is indeed a trivial example, see Definition 6.1.

Definition 6.3. The dual rational curve in $\mathbb{P}(S^n(\mathbf{E}_n^\vee))$ is denoted by $\mathcal{R}_n^\vee$.

$$\mathcal{R}_n^\vee =^{\text{def}} \{)\mathfrak{a}^n(|)\mathfrak{a}(\in \mathbb{P}_1^\vee\} \subset \mathbb{P}(S^n(\mathbf{E}_2^\vee)). \tag{6.5}$$

Since the duality $S^n(\mathbf{E}_2^\vee \times S^n(\mathbf{E}_2^\vee) \to \mathbb{C}$ induced functorially by the canonical one $\langle , \rangle : \mathbf{E}_2^\vee \times \mathbf{E}_2 \to \mathbb{C}$ already defined leads naturally to the incidence relation between points and hyperplanes in the ambient space $\mathbb{P}(S^n(\mathbf{E}_2))$ of $\mathcal{R}_n$ we look for a relation between $\mathcal{R}_n$ and $\mathcal{R}_n^\vee$.[4] We see that if $A = (\mathbf{a}) =)\alpha($ then $)\alpha^n($ *is the osculating hyperplane to* $\mathcal{R}_n$ in the point $(\mathbf{a}^n) = nA$ of $\mathcal{R}_n$ as an immediate consequence of $\langle \alpha^n, \mathbf{a}^n \rangle = 0$. If $n = 1$ the trivial polarity Π_1, see Definition 6.1, maps every point on the line in itself regarded as a hyperplane. If $n = 2$ we find that $\mathcal{R}_2^\vee$ is the envelope of tangent lines to the conic $\mathcal{R}_2$; $\mathcal{R}_3^\vee$ is the family of osculating planes to $\mathcal{R}_3$ in $\mathbb{P}_3$.

Let us consider next a digression on osculating spaces.

7. Osculating spaces to $\mathcal{R}_n$

The explicit expression of the duality is given by:

$$\langle \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n, \, \mathbb{b}_1 \mathbb{b}_2 \dots \, \mathbb{b}_n, \rangle = \frac{\operatorname{per}(\langle \mathbf{a}_i, \mathbb{b}_j \rangle)}{n!}$$

[4] We recall that the two natural monomial basis induced by the dual ones $\mathbf{x}, \mathbf{u}$ are orthogonal but not orthonormal. We do not normalize by dividing by obvious $\surd$'s without good geometrical meaning. We keep the $\mathbf{x}$ monomial basis but we replace $\mathbf{u}^h \mathbf{u}^{N-h}$ by the Taylor expansion-derivative $(h!(N-h)!)^{-1} \mathbf{u}^h \mathbf{u}^{N-h}$ since we have:

$$(h!(N-h)!)^{-1} \frac{\partial^N}{\partial^h \mathbf{x}_0 \partial^{N-h} \mathbf{x}_1} (\mathbf{x}_0^h \mathbf{x}_1^{N-h}) = 1$$

and $\langle \mathbf{a}, X \rangle = \sum a_I X^I$ and

$$\left\langle \sum a_H \mathbf{x}_0^h \mathbf{x}_1^{N-h}, \mathbf{b}^n / n! \right\rangle = \sum a_h b_0^h b_1^{N-h}.$$

where *per* denotes the *permanent* of the matrix $(\langle \mathbf{a}_i, \mathbb{b}_j \rangle)$ obtained from the determinant definition by replacing everywhere ± 1 by a universal 1 for any permutation. From this property it follows

$$\langle F, \mathbf{x}^n \rangle) = F(\mathbf{x}) \qquad \forall F \in S^n(\mathbf{E}_2^\vee) \qquad \forall \mathbf{x} \in \mathbf{E}_2$$

As a consequence we have:

1. The equation $\langle \mathbf{a}_1\ \mathbf{a}_2 \dots \mathbf{a}_n, \mathbf{x}^n \rangle = 0$ has n different roots $(\mathbf{a}_j^n)$ if the $(\mathbf{a}_j)$ are all distinct.
2. If $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}$ the equation $\langle\ \mathbf{a}^2\ \mathbb{b}_2\ \mathbb{b}_3 \dots\ \mathbb{b}_n, \mathbf{x}^n \rangle = 0$ has a double root provided no $\mathbb{b}_j$ is proportional to $\mathbf{a}$,..., etc. From these properties it follows that $\langle\ \mathbf{a}^m\ \mathbb{b}_{m+1}\ \mathbb{b}_{m+2} \dots\ \mathbb{b}_n,\ \mathbf{x}^n \rangle = 0$ represents for variable $\mathbb{b}$'s the $(n - m - 1)$-dimensional hyperplane bundle containing the space $\mathbb{T}_{m-1}(\mathbf{a}^n)$ with an "m-point-contact" at $(\mathbf{a}^n)$ $\Pi_{n-1}(\mathbf{a}^n)$ is the $(m-1)$-dimensional osculating space at the point $(\mathbf{a}^n)$ of $\mathcal{R}_n$, in particular for $n = 2, 3, \dots$ the tangent line, the osculating plane ... of $\mathcal{R}_n$ at $(\mathbf{a}^n)$. Dually we have $\mathbb{T}_{m-1}(\mathrm{A})$ is represented by the set of points

$$(\mathbf{a}^{n-m+1}, \mathbf{x}_1, \mathbf{x}_2 \dots \mathbf{x}_{m-1})$$

because we have:

$$\langle \mathbf{a}^m \mathbb{b}_1 \mathbb{b}_2 \dots \mathbb{b}_{n-m}, \mathbf{a}^{n-m+1} \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_{n-m}, \rangle = 0$$

In other words:

> The equations with multiple roots in $\mathbb{P}_1$ lead naturally to the osculating spaces of all the possible dimensions $0, 1, \dots, n-1$, e.g., the hyperplane $)\mathbf{a}^2 \mathbb{b}_3 \dots \mathbb{b}_n($ meets $(\mathbf{a}^n)$ with multiplicity ≥ 2 and precisely with multiplicity equal to two iff all the $)\mathbb{b}($ are different from $)\mathbf{a}($. More generally: $)\mathbf{a}^h\ \mathbb{b}_{h+1} \dots\ \mathbb{b}_n($ meets $(\mathbf{a}^n)$ with multiplicity h if no $)\mathbb{b}_j($ equals $)\mathbf{a}($ for $0 \leq h \leq n$.

The non-singular polarity Π_N in $\mathbb{P}_N$ maps a subspace S_h in a subspace S_{N-h-1}, accordingly the points or hyperplanes) contained in (resp. passing through) S_h are mapped into hyperplanes containing (into points of) S_{N-h-1}.

In particular an osculating space $\mathbb{T}_h(P)$ of $\mathcal{R}_N$ maps to an osculating $\mathbb{T}_{N-h-1}(P)$ of $\mathcal{R}_N$ at the same point P. In the case $N = 2n - 1$ every osculating space of dimension $n - 1$ of $\mathcal{R}_N$ is invariant by Π_N.

We shall characterize osculating spaces to $\mathcal{R}_N$ in terms of the divisors $\mathbf{D}_N$ of the projective line by means of the following lemma.

Lemma 7.1. *The hyperplane bundle through the osculating space* $\mathbb{T}_r(\mathbf{a}^N)$ *of* $\mathcal{R}_N$ *at the point* $(\mathbf{a}^N)$ *is the space* $\{\mathbf{a}^{r+1} \mathbb{b}_{r+2} \dots \mathbb{b}_n\}$, $\forall \mathbb{b}_j \in \mathbb{P}_1^\vee$, $j = r+2, r+2, \dots, n$. *Accordingly* $\mathbb{T}_r(\mathbf{a}^N)$ *is the* r*-dimensional subspace of* $\mathbb{P}(S^N(\mathbf{E}_2))$ *representing all the divisors of* $\mathbb{P}_1$ *of type* $(N - r)A + \mathbf{D}_r = (\mathbf{a}^{N-r} \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_r)$ *for all* $(\mathbf{x}_j) \in \mathbb{P}_1$, $j = 1, 2, \dots, r$.

Proof. It suffices to check that $\langle \mathbf{a}^{N-r}\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_r, \mathbf{a}^{r+1}\mathbb{b}_{r+2}\dots\mathbb{b}_n\rangle = 0$ for arbitrary choices of the $\mathbf{x}$ and $\mathbb{b}$. A differential geometric approach can be sketched also with the following limit considerations:

$$\operatorname*{Lim}_{\lambda\to 0}\left(\frac{(\mathbf{a}+\lambda\mathbf{x})^N - \mathbf{a}^N}{\lambda}\right) = (n\mathbf{a}^{n-1}\mathbf{x}) = (\mathbf{a}^{N-1}\mathbf{x})$$

for $\forall\mathbf{x}\in\mathbf{E}_2$ thus the result is trivially true for $r=1$. Let us assume we proved the result up the value $r<N$. We have similarly

$$\operatorname*{Lim}_{\lambda\to 0}\left(\frac{\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_r(\mathbf{a}+\lambda\mathbf{x}_r)^r - \mathbf{a}^r}{\lambda}\right) = (\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_r\mathbf{x}_{x+1}\mathbf{a}^{r-1})$$

for arbitrary choices of the $\mathbf{x}_j$. □

Example. In the case r=1 the tangent line to $\mathcal{R}_N$ at its point $(\mathbf{a}^N)$ is the locus:

$$\{(\mathbf{a}^{N-1}\mathbf{x});(\mathbf{x})\in\mathbb{P}_1\},$$

e.g., the tangent line to the conic $\mathcal{R}_2$ in $(\mathbf{a}^2)$ is the line $\{(\mathbf{ax});(\mathbf{x})\in\mathbb{P}_1\}$. We shall consider in §7 the osculating spaces to $\mathcal{R}_n$ at its point at infinity $n\mathbf{I}_\infty$, $\mathbf{I}_n\in\mathbb{P}_1$. In particular it is clear that $)\mathbf{a}^n($ is the osculating hyperplane to $\mathcal{R}_n$ at $(\mathbf{a}^n)$.

A similar treatment of the osculating spaces to $\mathcal{R}_N$ needed to studying the divisors on the curve is done in [SpT1] and [SpT2] as an approach to certain vector bundles.

8. Updating Clifford's theorem in terms of divisors in the line

Let us come back now with the problem of characterizing the bijective map:

$$(\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n)\longleftrightarrow)\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n(\tag{8.1}$$

coming from $A_j=(\mathbf{a}_j)=)\mathbf{a}_j($, $j=1,2,\dots,n$, see §5, and in particular

$$(\mathbf{a}^n)\longleftrightarrow)\mathbf{a}^n(\tag{8.2}$$

The solution was given essentially in the 19th century by the quoted Clifford's theorem (see [C1, BE, G1]). We update his result as follows:

1. *The two images of the same positive divisor* $A_1+A_2+\dots+A_n(=\tilde{A}_1+\tilde{A}_2+\dots+\tilde{A}_n$ *in* $\mathbb{P}_1^\vee)$*, see* §5*: the point* $(\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n)$ *and the hyperplane* $)\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n($ *of* $\mathbb{P}_n$ *correspond in a non-singular polarity:* $\Pi_n:\mathbb{P}_n\to\mathbb{P}_n^\vee$:

$$)\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n(= \textstyle\prod_n(\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n) \tag{8.3}$$

invariant by the projective group $\mathbb{P}GL(\mathbf{E}_2)$ *on the line acting on* $\mathbb{P}(S^n(\mathbf{E}_2))$ [BE, G1, H-P]. In particular: *The osculating hyperplane to* $\mathcal{R}_n$ *at a point* $(\mathbf{a}^n)$ *is* $)\mathbf{a}^n($ *for any* $A=(\mathbf{a})=)\mathbf{a}(\in\mathbb{P}_1$:

$$)\mathbf{a}^n(=\textstyle\prod_n(\mathbf{a}^n). \tag{8.4}$$

In other words: $)\mathbf{a}^n($ *describes the dual rational normal curve* $\mathcal{R}_N^\vee=\{)\mathbf{a}^n(|\mathbf{a}\in\mathbf{E}_2^\vee-0\}$.

2. $\Pi_{2\mu}$ *is the polarity with respect to a quadric* $Q_{2\mu}=0$, *containing* $\mathcal{R}_{2\mu}$ *when* $n=2\mu$ *is even. The tangent hyperplane to* $Q_{2\mu}$ *at a point* P *of* $\mathcal{R}_{2\mu}$ *is its osculating hyperplane at* P.
3. $\Pi_{2\nu+1}$ *is a Nullpolarity* (see [BE], [H-P], [G1]) *if and only if* $n=2\nu+1$ *is an odd integer. Then every point of* $\mathbb{P}_n$ *belongs to its polar hyperplane;* Π_n *reduces to* Π_1 *for* $n=1$.

Proof. In a sketchy manner, we update the original one checking that this Theorem is trivially true for $n=1$ (see the Π_1 of Definition 6.1) according to (9.8). Besides Π_1 generates Π_n because the monomial basis $S^n(\mathbf{u}_0,\mathbf{u}_1)$ is generated by $\mathbf{u}_0,\mathbf{u}_1$ and we have

$$\mathbf{u}_0\mapsto x_1,\mathbf{u}_1\mapsto -x_0\Rightarrow \mathbf{u}_0^j\mathbf{u}^{n-j}\mapsto(-1)^j\binom{n}{j}x_1^jx_0^{n-j}$$

and the $(n+1)\times(n+1)$ matrix representing (8.3) defined up to a non zero constant factor can be chosen equal to

$$\mathbb{M}_{n+1}=2^{\text{nd}}\,\text{diag}\left(1,-n,\binom{n}{2},\dots,(-1)^n\right)$$

with only zeros outside the secondary diagonal. We see that $\mathbb{M}_{n+1}=\pm{}^t\mathbb{M}_{n+1}$, and non singular, i.e., it is either antisymmetric if n is odd or symmetric if n is even. It is clear that the hyperplane $)\mathbf{a}^n($ cuts $\mathcal{R}_n$ in $(\mathbf{a}^n)$ since $\langle\mathbf{a},\mathbf{a}\rangle=0$. □

Remark 8.1. A reader acquainted with the symbolic method of invariant theory will recognize that *the left-hand side of the equation of the Clifford quadric* $Q_{2\mu}$ *is represented symbolically by* $(a,b)^n$ *if* a_x^n,b_x^n *are two equivalent symbolic representations of the binary form* $\mathbf{f}$ *of degree* n, *if* a *and* b *are not equivalent the antisymmetric invariant* $(a,b)^n$ *vanishes identically for* $a=b$ (see [W] or [G1]) if n is an odd integer (see the recent translation of [H]). We hope to come back to this approach in another forthcoming paper.

8.1. A pictorial description for $n=2,3$

We shall indicate a point or hyperplane in $\mathbb{P}(S^n(\mathbf{E}_2))$ with the same notation that the corresponding divisor in $\mathbb{P}_1$ (or $\mathbb{P}_1^\vee$) for instance $X+Y,2Z$ for points (or $\mathcal{X}+\mathcal{Y},2\mathcal{Z}$ for lines) in the plane.

The line $\mathcal{X}+\mathcal{Y}$ meets the conic $\mathcal{R}_2$ in the two distinct points $2X$ and $2Y$. Their tangents $2\mathcal{X}$ and $2\mathcal{Y}$ meet at the pole $X+Y$ of $\mathcal{X}+\mathcal{Y}$ with respect to Clifford's conic $Q_2=\mathcal{R}_2$ (see Figure 1). In $\mathbb{P}_3$ we have $X+Y+Z,2X+Z,3Z$ (or $\mathcal{X}+\mathcal{Y}+\mathcal{Z}$, $2\mathcal{X}+\mathcal{Z}$, $3\mathcal{Z}$ for planes); the three osculating planes at the three distinct points $3X,3Y,3Z$, meet in $X+Y+Z$ which lies in the plane $\mathcal{X}+\mathcal{Y}+\mathcal{Z}$ joining $3X,3Y,3Z$.

In the general case the hyperplane $)\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_n(=\tilde{A}_1+\tilde{A}_2+\dots+\tilde{A}_n$ in $\mathbb{P}_n$ image of a divisor $A_1+A_2+\dots+A_n$ in $\mathbb{P}_1$ meets $\mathcal{R}_n$ in the n image points $nA_j=(\mathbf{a}_j^n)$. Dually we can draw precisely the n osculating hyperplanes $)\mathbf{a}_j^n($ to $\mathcal{R}_n$ through the point $A_1+A_2+\dots+A_n$ of $\mathbb{P}_n$.

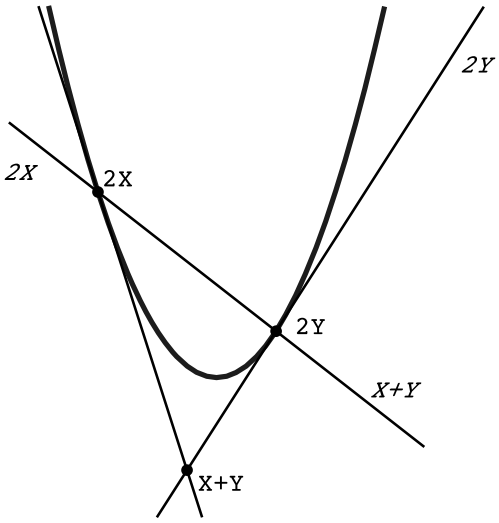

FIGURE 1. The pictorial description.

9. Affine properties of the $\mathcal{R}_n$

The $\mathcal{R}_n$ represented canonically has a unique point at infinity $\mathbf{I}(= n\mathbf{I}_\infty)$ of homogeneous coordinates $(0, \dots, 0, 1)$ image of $\mathbf{I}_\infty \in \mathbb{P}_1$ and the hyperplane at infinity $X_0 = 0$ osculates $\mathcal{R}_n$ in $\mathrm{n}\mathbf{I}_\infty$. We say that $\mathcal{R}_n$ has the parabolic type because in particular $\mathcal{R}_2$ is a parabola with the line at infinity tangent to $\mathcal{R}_2$ in its unique point at infinity $2\mathbf{I}_\infty$ and $\mathcal{R}_3$ is a twisted cubic with the plane at infinity osculating $\mathcal{R}_3$ in $(0, 0, 0, 1)$.

9.1. Projections of $\mathcal{R}_n$

The projection of $\mathcal{R}_n$ from an outside point $P \in \mathbb{P}_n - \mathcal{R}_n$ over a hyperplane H is an R_{n-1} and similarly projecting $\mathcal{R}_n$ from $n - h(> 0)$ generic points $P_1, P_2, \dots, P_{n-h}$ of $\mathcal{R}_n$ over $\mathbb{P}_h$ we obtain an $\mathcal{R}_h, s$, [B].

We shall apply this well-known property when $P_1 = P_2 = \cdots = P_{n-h} = \mathbf{I} =$ point at infinity of $\mathcal{R}_n$ and $\mathcal{R}_h$ is a coordinate space. Then we have the more precise.

Lemma 9.1. *The locus of* $(1, r, r^2, \dots, r^h, 0, 0, \dots, 0)$ *in* $\mathbb{P}_n$ (*with* $n - h$ *zeroes at the end*) *is a parabolic rational normal curve* $\mathcal{R}_h$ *in the coordinate space* $X_n = 0\, X_{n-1} = 0 \dots X_{h+1} = 0$ *projection of* $\mathcal{R}_n$ *from the osculating space* $\mathbb{T}_{n-h-1}(\mathbf{I})$.

Proof. It suffices to start with $h = n - 1$. The three points $(1, r, r^2, \dots, r^{n-1}, 0)$, $(1, r, r^2, \dots, r^n)$, $(0, 0, \dots, 0, 1)$, are collinear, thus the projection $(1, r, r^2, \dots, r^{n-1}, 0)$ of $(1, r, \dots, rn)$ from $n\mathbf{I}$ describes a canonical $\mathcal{R}_{n-1}$ of the parabolic type in the hyperplane $\chi_n = 0$ with point at infinity $(0, 0, \dots, 1, 0)$.

Iterating for $n - 2, n - 3, \dots, h + 1$ we prove the Lemma. □

Definition 9.2. We call $\mathcal{R}_{n-1}, \mathcal{R}_{n-2}, \ldots, \mathcal{R}_1$ the **I** projections of $\mathcal{R}_n$, for short.

Some of the properties proved in the classical theory of symmetric functions are invariant by these projections. Let us check as an example the well-known relations between the generating functions $e_1, e_2, \ldots, e_n$ and the $h_1, h_2, \ldots, h_n$. They show that these points are conjugate with respect tot he Clifford polarity in $\mathbb{P}_n$ the two projections $(e_1, e_2, \ldots, e_j)$, $(h_1, h_2, \ldots, h_j)$, $(j = n-1, n-2, \ldots)$ are also conjugate with respect to the Clifford polarity in $\mathbb{P}_j$.

In order to recover properties of the symmetric functions we are going to use the coordinates assuming that the $\mathcal{R}_n$ are always represented canonically as usual.

The conjugation condition of two generic points with coordinates $(e_1, e_2, \ldots, e_n)$, $(h_1, h_2, \ldots, h_n)$ with respect to the Clifford polarity is

$$\sum_{j=0}^{n} (-1)^j e_j h_{n-j} = 0 \quad e_0 = h_0 = 1 \tag{9.1}$$

We recognize in (9.1) the definition condition of the h functions (see [Mcd]) if we add

$$\sum_{j=0}^{m} (-1)^j e_j h_{n-j} = 0 \tag{9.2}$$

(for $m = n-1, \ldots, 2, 1$) but the condition for $m = n-1$ means that the two points $(e_1, e_2, \ldots, e_n)$, $(h_1, h_2, \ldots, h_n)$ from $n\mathbf{I}_\infty$ projections on $\mathcal{R}_{n-1}$ are conjugate with respect to $\mathcal{R}_{n-1}$ and by iteration $(e_1, e_2, \ldots, e_m)$ and $(h_1, h_2, \ldots, h_m)$ are conjugate with respect to $\mathcal{R}_m$ $(m = n-1, n-2, \ldots, 2, 1)$.

Notice that the property : $\mathcal{R}_{n-1}, \mathcal{R}_{n-2}, \ldots$ are the projections of $\mathcal{R}_n$ from the osculating space at infinity $T_0, T_1, \ldots$ (is equivalent in the algebraic Geometry of $\mathcal{R}_n$ to the construction of the images of the complete linear systems $(D_m - I_\infty, D_m - 2I_\infty, \ldots)$. In other words we have:

There is a unique point $(h_1, h_2, \ldots, h_n)$ *Clifford conjugate of* $(e_1, e_2, \ldots, e_n)$, *such that their projections*

$$[(e_1, e_2, \ldots, e_{n-1}), (h_1, h_2, \ldots, h_{n-1})],$$
$$[(e_1, e_2, \ldots, e_{n-2}), (h_1, h_2, \ldots, h_{n-2})], \ldots$$

from

$$T_1(nI_\infty), T_2(nI_\infty), \ldots$$

remain conjugate with respect to $\mathcal{R}_{n-1}$, $\mathcal{R}_{n-2}, \ldots$.

10. Agreement with Macdonald's approach

It is well known that do not exist global holomorphic functions on $\mathbb{P}_n$ except constants. Accordingly the classical theory of symmetric functions was developed over the affine line $\mathbb{A}_1 = \mathbb{P}(\mathbf{E}_2) - \mathbf{I}_\infty$ constructed with an arbitrarily prefixed $\mathbf{I} = \mathbf{I}_\infty$, see Def. 5.1. However, we cannot ignore that the coefficient a_n of a moving binary form f_n (see Definition 2.1) of degree n might become equal to

zero $\Leftrightarrow$ some root becomes $\mathbf{I}_\infty$. This property generates a stratification of the complete linear system $|\mathbf{D}_n|$ where we emphasize the growing multiplicity m of the undesirable $\mathbf{I}_\infty$:

$$\mathbf{D}_n = (n-m)\mathbf{I}_\infty + \mathbf{A}_m.$$

The flag of osculating spaces to $\mathcal{R}_n$ at $\mathbf{I}_\infty$:

$$n\mathbf{I}_\infty = \mathbb{T}_0(\infty) \subset \mathbb{T}_1(\infty) \subset \cdots \subset \mathbb{T}_{n-1}(\infty) \tag{10.1}$$

describes the degeneration of such a moving divisor $\mathbf{D}_n$ in terms of its maximal affine divisor $\mathbf{A}_m \in \mathbb{A}_1^{(m)}$, $m = n, n-1, \ldots, 2, 1, 0$ (see Definition 2.3) when the multiplicity of $\mathbf{I}_\infty$ grows one by one. $\mathbf{A}_m$ is represented by a monic polynomial F_m of degree $m \leq n$.

We want to study (10.1), see [G2], as a geometric object – where the actual symmetric functions live – justifying the [Mcd] stratification Λ_n of the ring Λ of the so called "symmetric functions" in infinitely many variables according to [Mcd] in such a way that Macdonald's Λ_n becomes the ring of symmetric functions in some $\mathbb{A}_n$ model of $\mathbb{A}_1^{(n)}$ keeping the geometric properties and the functional essence and the beauty of the formal approach as much as possible. On the other hand *the constant and successful use of the generating functions should be compatible with the explicit study of the rational normal curves* $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$, done implicitly via the sequence of powers $1, x, x^2, \ldots, x^n, \ldots$ which has a natural extension to several variables via Veronese varieties.

We quote literally [Mcd], page 18 to convince the reader of how reasonable seems the compromise developed in [G2]:

> "*In the theory of symmetric functions the number of variables is usually irrelevant, provided only that it is large enough, and it is often more convenient to work with infinitely many variables ...*"
> "*To make this idea precise, let $m \geq n$ and consider the homomorphism $\mathbb{Z}[x_1, x_2, \ldots, x_m] \to \mathbb{Z}[x_1, x_2, \ldots, x_n]$ which sends each of x_{m+1}, x_{m+2}, ..., x_n to zero and the other x_i to themselves ...*"

In this paper we do not want to forget that the x_j come from the roots $r_1, r_2, \ldots, r_N$ of a moving equation $F = 0$. If the leading coefficient vanishes one of the roots becomes $\mathbf{I}_\infty$. Combinatorialists change F by the generating polynomial g (see Definition 9.1, page 17) for well-known reasons. The roots of g equal $-r_j^{-1}$ thus $r \to \infty$ implies $-r^{-1} \to 0$. Thus our approach is compatible with Macdonald's. We shall see what happens when one,two, ..., the n roots become infinite. Notice that $-r^{-1}$ is the affine hyperplane coordinate corresponding to the point $r \neq 0$.

Let us come back to our previous notations assuming $N \gg n$. A generic hyperplane of $\mathbb{P}_N$ meets $\mathcal{R}_N$ in N distinct affine points corresponding to roots $r_1, r_2, \ldots, r_N$ ($\neq 0$ and ∞).

The initial given polynomial of degree n defines an affine divisors of roots $\mathbb{A}_n$ not containing 0 in the constant term is $\neq 0$. The corresponding hyperplane in $\mathbb{P}_N$ cuts $\mathcal{R}_N$ at the point at infinity with multiplicity $N - n$.

The notation

$$1 + e_1 x + e_2 x^2 + \cdots = (1 - \rho_1 x)(1 - \rho_2 x) \ldots$$

where $\rho_j = -\frac{1}{r_j}$, $r_j \neq 0$, for all j, expresses the e_j as functions of (ρ_1, ρ_2, …, $\rho_n, 0, 0, \ldots, 0$) where the zeros come from the $N - n$ infinite roots.

The traditional notations

$$(x - r_1)(x - r_2) \cdots = x^n - e_1 x^{n-1} + e_2 x^{n-2} + \cdots$$

assume $e_j = e_j(r_1, r_2, \ldots, r_n)$.

The change of notations reflects Clifford theorem changing points $(e_0 + re_1)$ by hyperplanes $)x_1 - rx_0($ in the projective line; since the Clifford polarity permutes 0 and ∞ and r, $\rho = -\frac{1}{r}$ with $r \neq 0$ and ∞.

We write the following disjoint union of affine spaces:

$$\mathbb{P}_N = \mathbb{A}_1^{(N)} \cup (\mathbb{A}_1^{(N-1)} + \mathbf{I}) \cup (\mathbb{A}_1^{N-2} + 2\mathbf{I}) \cup \cdots \cup (N\mathbf{I}) \tag{10.2}$$

in agreement with the coefficient sequences:

$$(a_0, a_1, \ldots a_N),\ (a_0, a_1, \ldots a_{N-1}) \ldots, (a_0).$$

(10.2) is a particular case of the disjoint union decomposition

$$\mathbb{P}_N = \mathbb{A}_0 \cup \mathbb{A}_1 \cup \cdots \cup \mathbb{A}_{N-1} \cup \mathbb{A}_N \tag{10.3}$$

(where $\mathbb{A}_i \cap \mathbb{A}_j = \emptyset$ for $i \neq j$) of a projective space $\mathbb{P}_N$ in terms of a prefixed maximal flag

$$\mathbb{P}_0 \subset \mathbb{P}_1 \subset \cdots \subset \mathbb{P}_{N-1} \subset \mathbb{P}_N,$$

where $\mathbb{A}_0 = \mathbb{P}_0$ and $\mathbb{A}_r = \mathbb{P}_r - \mathbb{P}_{r-1}$, $r > 0$. (10.3) can be extended to a $\mathbb{P}_\infty$ as follows:

$$\mathbb{P}_\infty = \mathbb{A}_0 \cup \mathbb{A}_1 \cup \cdots \cup \mathbb{A}_n \cup \cdots \tag{10.4}$$

Indeed in our case $\mathbb{P}_N = \mathbb{P}(S^N(\mathbf{E}_2))$ and the flag consists of the osculating spaces to $\mathcal{R}_N$ at infinity, where $N \geq n$.

Part II. Grassmannnian Geometry Applied to the Variety $\Sigma(n; N)$

We only need a few prerequisites on $\mathcal{G} = \mathcal{G}(n-1; N)$ according to [Sch].

11. Ubiquity of the Schur functions and the F-coefficients via $\mathcal{R}_N$

Let us consider again the case of a $\mathbf{D}_n$ of n distinct points lying on an $\mathcal{R}_N$ with a fixed $N \geq n$. Even for $N = n$ we see that the e_j are Schur functions by solving with Cramer's rule a system of equations for the e's with determinant $V \neq 0$. *For any $N \geq n$ the Schur functions reappear everywhere in $\mathbb{P}_N$ as point-Grassmann coordinates, of the $(n-1)$-dimensional subspace $S(\mathbf{D}_n)$ of $\mathbb{P}_N$ span by $\mathbf{D}_n$*, as well as the coefficients of f or F, see Lemma 11.3 below, reappearing in the staircase form matrices (11.3) of Lemma 11.3 define the dual Grassmann coordinates of $S(\mathbf{D}_n)$.

Lemma 11.1. *The $(n-1)$-dimensional space $S(\mathbf{D}_n)$ n-secant of $\mathcal{R}_n$ is well defined for any* $\mathbf{D}_n = m_1P_1 + m_2P_2 + \cdots + m_hP_h$, *with $P_i \neq P_j$ for $i \neq j$, $m_i > 0$ for all i and $m_1 + m_2 + \cdots + m_h = n$.*

Proof. Indeed the hyperplane sections $\mathbf{H}.\mathcal{R}_N = \mathbf{D}_n + \mathbf{E}_{N-n}$ containing $\mathbf{D}_n$ describe outside $\mathbf{D}_n$ the complete linear system $\mathbf{E}_{N-n}$ thus the $\mathbf{H}$'s intersect in a subspace S_{n-1} containing the *osculating $\mathbb{T}_{m_j-i}$ space of dimension $m_j - 1$* at the point P_j of multiplicity m_j in $\mathbf{D}_n$, but it does not contain $\mathbb{T}_{m_i}$. Since we need explicit formulas we split this argument in Lemmas 11.1 and 11.3 by extending the definition of $S(\mathbf{D}_n)$ when there are multiple points or when it is defined as an intersection of $N - n + 1$ linearly independent hyperplanes. If $F = 0$ has at least one multiple root all the $n \times m$ minors of $\mathbb{R}$ in (2.1) vanish.

We modify the definition of $\mathbb{R}$ as follows: let $\mathbb{R}$ be now the $n \times (N+1)$ matrix built by h vertical juxtapositions of the h matrices $\mathbb{M}_i$ of size $m_i \times (N+1)$ with the coordinates of the ith point P_i of the support of $\mathbf{D}_n$ and the derivatives of order $1, \ldots, m_j - 1$, for $j = 1, 2 \ldots, h$:

$$\mathbb{M}_i = \begin{pmatrix} 1 & r_1 & r_1^2 & \ldots & r_1^{N-1} & \ldots & r_1^N \\ 0 & 1 & 2r_1 & \ldots & (N-1)r_1^{N-2} & \ldots & Nr_1^{N-1} \\ 0 & 0 & 2 & \ldots & (N-1)(N-2)r_1^{N-3} & \ldots & N(N-1)r_1^{N-2} \\ \cdot & \cdot & \cdot & \ldots & \cdot & \ldots & \cdot \\ 1 & r_2 & r_2^2 & \ldots & r_2^{N-1} & \ldots & r_2^N \\ \cdot & \cdot & \cdot & \ldots & \cdot & \ldots & \cdot \end{pmatrix} \tag{11.1}$$

The point-Grassmann coordinates of $S(\mathbf{D}_n)$ are the $n \times n$ minors extracted from the matrix $\mathbb{R}$ of maximal rank:

$$\mathbb{R} = \begin{pmatrix} \mathbb{M}_1 \\ \mathbb{M}_2 \\ \vdots \\ \mathbb{M}_h \end{pmatrix} \tag{11.2}$$

The condition

$$\sum a_i X_i = 0$$

is satisfied if we replace X_i by the derivative of order $0, 1, \ldots, m_j - 1$ computed in r_j for $j = 1, 2, \ldots, h$. Conversely any polynomial degree n satisfying these conditions has the type $\lambda F \Rightarrow \operatorname{rank} \mathcal{R} = n$. As a consequence the space $S(\mathbf{D}_n)$ contains each one of the h support points P_j but it contains also the shown osculating space $\mathbb{T}_{m_j - 1}$ at P_j (but not the next one $\mathbb{T}_{m_j}$). □

Definition 11.2. Equivalently we say that $\mathbb{T}_{m_j - 1}$ has an "*m_j-point contact*" with $\mathcal{R}_n$. If $h = n$ and $m_j = 1$, for all j, and the matrix $\mathbb{R}$ becomes again (2.1).

Lemma 11.3. *The rows of the $(N - n + 1) \times (N + 1)$ matrix $\mathbb{A}$:*

$$\mathbb{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot & \ldots & a_{n-1} & 1 & & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & a_0 & a_1 & \cdot & a_{n-2} & a_{n-1} & 1 \end{pmatrix} \tag{11.3}$$

built with the coefficients of

$$F = \prod (x - r_j)^{m_j}$$

defining an affine divisor $\mathbf{A}_n$ represents $N+1-n$ linearly independent hyperplanes in $\mathbb{P}_N$ intersecting in the $S_{n-1}(\mathbf{A}_n)$. The $p_{\ldots}$ of the $S_{n-1}(\mathbf{A}_n)$ defined by $\mathbb{A}$ are normalized by the condition $p_{n,n+1,\ldots,N} = 1$.

Proof. The rows of $\mathbb{A}$ are the coefficients of the binary forms:

$$x_0^{N-n} f,\ x_0^{N-n-1} f x_1,\ x_0^{N-n-2} f x_1^2,\ \ldots,\ f x_1^{N-n} \tag{11.4}$$

representing the $N + 1 - n$ linearly independent N-divisors in $\mathbb{P}_1$:

$$\mathbf{D}_n + (N-n)\mathbf{I}_\infty, \mathbf{D}_n + (N-n-1)\mathbf{I}_\infty + \mathbf{O}, \ldots, \mathbf{D}_n + (N-n)\mathbf{O} \tag{11.5}$$

where f becomes F for $x = x_1/x_0$.

The coordinate $p_{n,n+1,\ldots,N}$ of the space S_{n-1} represented by $\mathbb{A}$ is equal to 1 since the rightmost minor is lower triangular with only 1-entries in the main diagonal. □

We removed all the restrictions on $\mathbf{D}_n$. Accordingly we have the following definition.

Definition 11.4. $S(\mathbf{D}_n)$ is the well-defined space of dimension $n - 1$ of $\mathbb{P}_N$ defined by the divisor $\mathbf{D}_n$ of $\mathcal{R}_N$ spanned by the osculating spaces of dimension $m_j - 1$ at every point of $\mathbf{D}_n$ counting with multiplicity m_j, for $j = 1, 2, \ldots, h$ (see the proof of Lemma 11.1).

Definition 11.5. *The locus of such $S(\mathbf{D}_n)$ is a well-defined algebraic rational subvariety $\sum(n; N)$ of dimension n inside the Grassmannian $\mathcal{G}(n-1; N) = \mathcal{G}(S_{n-1} \subset \mathbb{P}_N)$ (or $\mathcal{G}$ for short)*, (see a proof in Section 17), but we shall use from now on this terminology applied to the well-defined set $\{S(\mathbf{D}_n)\}$.

In order to establish this fact we need to recall a few properties of the Grassmannian:

1. *Proportionality relations between $p^{\cdots}$ and $p_{\cdot\cdot}$* (see [H-P]). We shall need them in the form:
$$\rho p^{i_0 i_1 \dots i_{n-1}} = \operatorname{sgn}(i,j) p^{j_n j_{n+1} \cdots j_N} \quad \rho \in \mathbb{C}^\times \tag{11.6}$$
where (i,j) is short for the permutation $(i_0, i_1, \dots, i_{n-1} j_n j_{n+1} \dots j_N)$ of $0, 1, \dots, N$ and $0 \le i_0 < i_1 < \cdots < i_{n-1} \le N$, $0 \le j_n < j_{n+1} < \cdots < j_N \le N$. We shall use often the normalizations
$$p^{01,\dots,n-1} = p_{n,n+1,\dots,N} = 1$$
inside $\mathcal{G}$ (see [BE, H-P]), belonging to well-defined *local charts* (see below). In this important particular case we choose $\rho = 1$ and the proportionality relations become identities relating both kinds of Grassmann coordinates $p^{\cdots}, p_{\dots}$ with complementary indices.
2. *Coordinate books.* The most natural *local representation* of a given Grassmannian $\mathcal{G} = \mathcal{G}(n-1; N)$ is obtained by the $n(N+1-n) = \dim \mathcal{G}$ coordinates of the n points $P_j = S.\Pi^{(j)}$ of intersection of a variable subspace $S_{n-1} \subset \mathbb{P}_n$ with n prefixed subspaces $\Pi^{(j)}$, of dimension $N-n+1$, *provided the P_j are linearly independent.*

Definition 11.6. We say that *the $\Pi^{(j)}$ are the n pages of a book with hinge* $\mathbf{H_{N-n}}$, if the $\Pi^{(j)}$ do not lie in a hyperplane and they contain a common subspace $\mathbf{H}_{N-n}$ of the shown dimension.

We can see easily that every S_{n-1} not meeting $\mathbf{H}$ cuts the $\Pi^{(j)}$ in n linearly independent points and conversely, e.g., a variable line ℓ in $\mathbb{P}_3$ is parametrized by the two distinct intersection points with two planes α, β *provided ℓ does not cut the hinge $\mathbf{H} = \alpha \cap \beta$ of the book with the two pages α, β.*

A fundamental example is given in the following definition: the $p^{01\cdot\cdot,n-1} \neq 0$ condition is equivalent with the property that S_{n-1} does not meet the coordinate $(N-n)$-hinge $\mathbf{H}_{n,n+1,\dots,N}$ of the book $\mathcal{B}_{01\dots,n-1}$ span by $\mathbf{H}$ and the vertices $\mathbf{0}, \mathbf{1}, \dots, \mathbf{n-1}$ of the chosen projective frame $\mathcal{S} = \{\mathbf{0\,1\,2} \dots, \mathbf{N}; \mathbf{U}\}$.

Definition 11.7. $\mathcal{S}$ induces affine coordinate frames in each affine space $\mathbb{A}^{(j)} = \Pi^{(j)} - \mathbf{H}$ where $\Pi^{(j)} = \mathbf{Hj}$ is an $(N-n+1)$-dimensional page of the coordinate book $\mathcal{B}_{0,1,\dots,n-1}$. $\Pi^{(j)}$ is a coordinate space joining $\mathbf{H}$ with the coordinate vertex for $\mathbf{j} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{n-1}$ (see [G1]).

Accordingly: $\mathcal{G}$ *is covered by an Atlas containing* $\binom{N+1}{n}$ *coordinate books.*

Let us assume $p^{01\dots n-1} \neq 0$. Let $\mathbf{M} \backsimeq F^{-1}M = (\mathbf{1}_n \mathbb{X}) \backsimeq (\mathbb{U}_n \mathbb{S})$ be equivalent $n \times (N-n+1)$ coordinate matrices representing the same S_{n-1} of $\mathcal{G}$ where F is the corresponding leftmost $n \times n$ minor of the matrix M, $\mathbf{1}_n$ is the shown unit matrix and $\mathbb{U}_n$ is defined by:
$$\mathbb{U}_n = 2^{\text{nd}} - \operatorname{diag}(1, -1, \dots, (-1)^n) \qquad \det \mathbb{U}_n = 1 \tag{11.7}$$

with only zeros outside the secondary diagonal which is alternating of type: 1, −1, 1, −1, ... Both $(\mathbf{1}_n\mathbb{X})$ and $(\mathbb{U}_n\mathbb{S})$ define the same absolute Grassmann coordinates of S normalized by $p^{01\cdots n-1}=1$. We shall use both of them related by $\mathbb{S}=\mathbb{U}_n\mathbb{X}$. The remaining normalized ones $p^{\cdots}$ become identical with the $k\times k$ minors of $\mathbb{S}$ (for all the possible cases $1\le k\le \min(n, N+1-n)$) up to the sign and the ordering (see [C-E-P]) but only the matrix $(\mathbb{U}_n\mathbb{X})$ has both nicest behaviours, namely: The formula:

$$p(0\dots\hat{i}_1\dots\hat{i}_2\dots\hat{i}_k\dots nj_1j_2\dots j_k)=s(i_1i_2\dots i_k;j_1j_2\dots j_k) \tag{11.8}$$

is obtained by dropping the i's rows from $0,1,\dots,n-1$ and adding the j's $(>n)$ columns (no sign ambiguity!), in particular for $k=1$ we have the following lemma.

Lemma 11.8. *The $n(N+1-n)=\dim\mathcal{G}$ entries of $\mathbb{S}$: $s_{ij}=p(\hat{i};j)$ read in Western reading order are local coordinates in $\mathcal{G}$ equal to the first lexicographically ordered $p^{\cdots}$ after the first one $p^{01\cdots n-1}=1$. Namely: $(s_{i_1},s_{i_2},\dots,s_{i_{N-n+1}})$ are the absolute coordinates of the intersection of S_{n-1} with the $(N-n+1)$-dimensional page joining the hinge $\mathbf{H}_{n+1,n+2,\dots,N}$ with $\mathbf{i}$ for $\mathbf{i}=\mathbf{0},\mathbf{1},\mathbf{2},\dots,\mathbf{n-1}$ (see Definition* 11.7).

We shall use the following identity

$$\mathbb{X}=-{}^t\mathbb{B} \tag{11.9}$$

for a couple of normalized matrices $\mathbb{R}\backsimeq(\mathbf{1}_n\mathbb{S}), \mathbb{A}\backsimeq(\mathbb{B}\mathbf{1}_{N-n+1})$. The normalized version of the incidence relations of type

$$\sum r_ja_j=0$$

between points and hyperplanes leads immediately to the relation (11.9) to be restricted in §12 from $\mathcal{G}$ to $\Sigma(n;N)$.

12. The variety $\Sigma(n;N)$ of n-secant spaces of $\mathcal{R}_N$ and its dual

In this section we shall restrict to the subvariety $\Sigma(n;N)$ (see Definition 11.2) well-known global properties of the Grassmannnian $\mathcal{G}(n-1;N)$. $\Sigma(n,N)$ was studied in [Sch] as an embedding $\mathbb{P}_1\hookrightarrow\mathcal{G}(n-1;N)$ defining a vector bundle of rank n over $\mathbb{P}_n$.

$\Sigma(n;N)$ is naturally parametrized by the homogeneous coefficients a_j of f (see Definition 2.1). In particular we can assume $a_n=1$ if and only if $a_n\neq 0$.

Definition 12.1. We call the **affine part of** Σ – the locus $\Sigma(n;N)$ of the $S(\mathbf{A}_n)$ span by the affine divisors $\mathbf{A}_n$ of $\mathbb{A}_1^{(n)}=\mathbb{A}_1^n/\mathcal{S}_n$. We recall that $\mathbb{A}_1=\mathbb{P}_1-\mathbf{I}_\infty$ is parametrized by the values $e_j(r_1,r_2,\dots,r_j)$ of the elementary symmetric functions of the roots r_j of $F=0$, representing $\mathbf{A}_n$.

The remaining Grassmann-Schur coordinates are all the possible $k\times k$ minors of $\mathbb{S}$ (see Part II and [C-E-P]). We are going to remove the restriction $V\neq 0$, essentially equivalent to the non-vanishing of the discriminant of F.

Lemma 12.2. *The condition* $p^{01\cdot\cdot,n-1} \neq 0$ *holds for an* $S(\mathbf{D_n})$ *if and only if the positive divisor* $\mathbf{D}_n$ *is affine* (*see Definition* 2.1).

Proof. If $\mathbf{D}$ is affine $p^{01\cdot\cdot,n-1} \neq 0$ because the defining form f (see Definition 2.2) can be normalized as a monic polynomial F since the matrix $\mathbb{A}$ of Lemma 11.3 has the property $p_{n,n+1,\ldots,N} = 1$ and this implies $p^{01\cdot\cdot,n-1} \neq 0$ after (11.6). If the point $n\mathbf{I}_\infty$ with homogeneous coordinates $(0,0,\ldots,0,1)$ belongs to $S(\mathbf{D})$ it is clear that $p^{01\ldots,n-1} = 0$. □

If $\mathbf{D}_n = \mathbf{I} + \mathbb{A}_{n-1}(\mathbb{A}_{n-1} > 0$, affine), the matrix $\mathbf{R}$ of $(0,1)$ has the form

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & r_2 & \ldots & r_2^{N-1} & r_2^N \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ 1 & r_n & \ldots & r_n^{N-1} & r_n^N \end{pmatrix}$$

We see that the Grassmann coordinates of $S(A_{N-1})$ equal certain Grassmann coordinates of $S(\mathbf{I} + \mathbb{A}_{n-1})$, namely

$$p^{i_0 i_1 \ldots i_{n-1}} = p^{i_0 i_1 \ldots i_{n-1} N} \qquad 0 \leq i_1 < i_2 < \cdots < i_{n-1} \leq N-1$$

which are also equal to the corresponding Grassmann coordinates of the **I**-projection $\widetilde{S}(A_{n-1})$ of $S(A_{n-1})$ in $\mathbb{R}_{N-1}$ where $\widetilde{S}(A_{n-1})$ is $(n-1)$-secant to $\mathcal{R}_{N-1}$.

Similar statements holds by iteration to the divisors of type $h\mathbf{I} + A_{n-h}$ and the projection $\widetilde{S}(A_{n-h})$, $(n-h)$-secant of $\mathbb{R}_{N-h}$.

12.1. The *h*-basis of the vector space $\mathbb{P}^{-1}(\mathbb{S}(\mathbf{A}_n))$

Another remarkable basis of the vector space defining $S(\mathbf{A}_n)$ consists of the n rows of the matrix (12.1) below constructed with the *complete homogeneous functions* $h_j, j = 1, 2 \ldots$, namely we have:

Lemma 12.3. *The* $n \times (N+1)$ *matrix*

$$\mathbb{H} = \begin{pmatrix} 1 & h_1 & h_2 & h_3 & & \ldots & & & h_N \\ 0 & 1 & h_1 & h_2 & & \ldots & & & h_{N-1} \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \ldots & h_1 & h_2 & \cdot & h_n \end{pmatrix} \tag{12.1}$$

constructed with the values $h_r(r_1, r_2, \ldots, r_n)$ *of the* h_r *is another basis normalized by* $p^{01\ldots,n-1} = 1$ *of the vector space* $\mathbb{P}^{-1}(S_{n-1})$.

Proof. The well-known relations

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \quad \forall n \geq 0 \tag{12.2}$$

imply that the n shown points satisfy the system of equations with matrix

$$\mathbf{E} = ((-1)^{i-j} e_{i-j}) \quad i = 1, 2, \ldots, N-n+1, \ j = 0, 1, \ldots, N \tag{12.3}$$

where we write $e_r = 0$ for $r < 0$. The rows of $\mathbb{E}$ represents $N - n + 1$ linearly independent hyperplanes through $S(\mathbf{A}_{n-1})$ since its leftmost $n \times n$ matrix is upper

unitriangular. We deduce easily another basis of $\mathbb{P}^{-1}(S(\mathbf{A}_n))$ of type $(\mathbf{1}\mathbb{S}')$ whose last row equals $(0\ 0\dots 1\ h_1\ h_2\dots h_{N-n+1})$. Left multiplication by $\mathbb{U}$ leads to the same $(\mathbb{U}\ \mathbb{S})$ coming from $\mathbb{R}$. $(\mathbf{1}\mathbb{S})$ has $(0\ 0\dots 1\ h_1\ h_2\dots h_{N-n+1})$ in the first row. □

We can construct a similar variety $\Sigma^\vee(n;N)$ (called the dual of $\Sigma(n;N)$) attaching to a divisor

$$\mathbf{D}_n = A_1 + A_2 + \cdots + A_n$$

the intersection $S^\vee_{N-n-1}$ of the osculating hyperplanes $\mathbb{T}_{N-1}(A_j)$, $j = 1, 2, \dots, n$. $S^\vee_{N-n-1}(\mathbf{D}_n)$ is also well defined for any $\mathbf{D}_n$ but we do not obtain a new theory since the locus $\{\mathbb{T}_{N-1}(P); P \in \mathcal{R}_N\}$ is another rational normal curve $\mathbb{R}^\vee_N$ in $\mathbb{P}^\vee_N$. See Part I, in particular Clifford theorem.

13. The Hook-Schur functions as local Grassmann coordinates in Σ

We shall compare now lemmas 11.1 and 11.3 by restricting from $\mathcal{G}$ to Σ, see Definition 2.3, the recalled proportionalities between the $p^{\cdots}, p_{\dots}$ with complementary indexes defining the same S_{n-1} in $\mathbb{P}_N$ (see Part II). Since $p^{01\cdots,n-1} = p_{n,n+1,\dots,N} = 1$ they become identities (up to the sign) between the Schur functions and the $p_{\dots}$ expressed in terms of the maximal minors of $\mathbb{A}$.

We shall apply the well-known Lemma 11.8 valid for the whole Grassmannnian $\mathcal{G}$ to the affine part $\Sigma(n;N)$ of $\Sigma(n;N)$ taking advantage of the Lemma 12.3.

Remark 13.1. *The* partition notation. *The $p^{i_0 i_1 \cdots i_{n-1}}, p_{\dots}$ of an $S_{n-1} \in \mathcal{G}$ might be denoted also by π^λ (or π_λ) where $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n, 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le N - n + 1$, denotes a partition and we define $\lambda_j = i_j - j$, $j = 0, 1, \dots, n$ in agreement with the combinatorialists. Notice that the* parts *of λ are written in ascending order.*

Definition 13.2. We call **Schur functions of hook type** those corresponding to a partition of type $(1^{i-1}j)$.

Lemma 13.3. *The local coordinates of a space $S(\mathbf{A}_n) \in \Sigma(n;N)$ are equal to the values $s_{ij}(r_1, r_2, \dots, r_n)$ of the shown Schur functions of the hook type $(1^{i-1}j)$. In particular the entries of the first column (row) are the e_j, (the h_j's), see Lemma* 12.3.

Proof. As a consequence of the recalled property of the local $p^{\cdots}$'s in $\mathcal{G}$ we have: Any Schur function (Grassmann coordinate of an S belonging to $\Sigma(n;N)$, see §0), is equal to some $k \times k$ minor of the $\mathbb{S}$ matrix provided $1 \le k \le \min(n, N+1-n)$ and conversely. In particular we find that *the local coordinates $s_{ij}(r)$ are equal to the shown values of the Schur functions of the hook type.* □

We shall express both kinds $p^{\cdots}$ and $p_{\dots}$ of coordinates of an $S_{n-1} \in \Sigma$ in terms of the e_j's.

Lemma 13.4. *The ith row of* $\mathbb{S}$ *describes the coordinates of the intersection of an* $S(\mathbf{A}_n)$ *with a well-defined coordinate subspace of dimension* $N-n+1$*, in particular the first row displays the values of* $h_j(r)$ *of the complete homogeneous symmetric functions for* $j = 1, 2, \ldots, N$*.*

Proof. It is a restriction of a global property of $\mathcal{G}$ to Σ, see introduction to Part II. The h statement comes from Lemma 13.3 and Lemma 12.3. □

Examples. Let us fix n and consider increasing values of $N = n, n+1, \ldots$. If $N = n$ the matrix $\mathbb{S}$ has only 1 column with entries $e_j(r)$. (writing down $n = 2$ as the simplest non trivial case). In general we have the following normalized matrix: $\mathbb{R} = (\mathbb{U}_n\ \mathbb{S}_{n\times(N-n+1)})$, where $e_1 = h_1$ and

$$\mathbb{S}_{n\times(N-n+1)} = \begin{pmatrix} e_1 & h_2 & h_3 & \ldots & h_N \\ e_2 & . & . & \ldots & . \\ . & . & . & \ldots & . \\ e_n & . & . & \ldots & . \end{pmatrix} \qquad e_1 = h_1 \tag{13.1}$$

The value of $N = 2n-1$ is particular interesting because the first row of $\mathbb{S}_{n\times n}$ contains the $\mathbb{Z}$-basis $(h_1, h_2, \ldots, h_n)$ of Λ_n (see [Mcd]). A geometrical explanation comes in terms of the variety Ω swept by the points of a variable secant space $S_{n-1} \in \Sigma$, (see §14) and from Clifford's theorem.

14. $\Sigma(n;2n-1)$ and a new geometrical meaning of the h functions

If $N \leq n+n-1$ we expect that the S's passing through a generic point outside $\mathcal{R}_n$ describe an algebraic variety of dimension $2n-1-N$, in particular if $N = 2n-1$ it should consist of finitely many points. In the case $n = 2$ it is well known that *there exists a unique chord of an* $\mathcal{R}_3$ *passing through any outside point.* Given a generic plane π there is a $(1-1)$ map of the points P of π to the corresponding chords of $\mathcal{R}_3$ with the only exceptions arising from the three intersection points of $\mathcal{R}_3$ with π. In this case the plane $\{\mathbf{012}\}$ osculates $\mathcal{R}_3$ at its ∞ point, thus the exceptions are only the chords joining ∞ with any other point of $\mathcal{R}_3$. In the general case we have:

Lemma 14.1. *The secant space* $S(\mathbf{A}_n)$ *of* $\mathcal{R}_{2n-1}$ *attached to the divisor* $\mathbf{A}_n$ *of the roots of nth degree equation* $F = 0$ *meets the coordinate affine space*

$$\{\mathbf{n-1}, \mathbf{n}, \ldots, \mathbf{2n-1}\} - \{\mathbf{n} \ldots \mathbf{2n-1}\}$$

in the point with local coordinates equal to $(h_1(r), h_2(r), \ldots, h_n(r))$*. Conversely: there exists a unique n-secant space* $S(\mathbf{A}_n)$ *passing through any point*

$$(h_1, h_2, \ldots, h_n) \in \{\mathbf{n-1}, \mathbf{n}, \ldots, \mathbf{2n-1}\} - \{\mathbf{n} \ldots \mathbf{2n-1}\}.$$

If $N > 2n-1$ the first row of the matrix $\mathbb{S}$ of the formula (13.1) contains also the functions $h_{n+1}, \ldots, h_N$ which are polynomial functions of the $h_1, h_2, \ldots, h_n$, i.e., the intersections of the variable $S(\mathbf{A})$ with the coordinate space $\mathbf{Hn}$ describe a rational affine variety of this space.

15. The jth column entries of $\mathbb{S}$ as coefficients of the remainder of $x^{n-1+j} \mod F$ and the power sums

We shall prove in this section that the jth column of the $n \times (N-n+1)$ matrix $\mathbb{S} = (s_{ij})$ displays *essentially* (up to ordering and signs) the coefficients of the well-defined polynomial R_j ($\equiv x^{n-1+j} \mod f$) of degree less than n, i.e., as usual we have:

Definition 15.1. We define R_j as the **remainder mod** F of x^{n-1+j} for $j = 1, 2, \ldots$.

In order to make a precise statement we shall replace the matrix $\mathbb{A}$ of Lemma 11.3 by the following $(-\mathbb{B}\ \mathbf{1}_{N-n+1})$ where $\mathbb{B} = (b_{jk})$ is the coefficient matrix of the R_j namely

$$R_j = b_{j0} + b_{j1}x + b_{j2}x^2 + \cdots + b_{j,n-1}x^{n-1} \equiv x^{n-1+j} \tag{15.1}$$

and $F = x^n - f_1$, i.e., we shall write:

$$\mathbb{A} \backsimeq (-\mathbb{B}\ \mathbf{1}_{N-n+1}) \quad \Rightarrow \quad \mathbf{R} \backsimeq (\mathbf{1}_n\ {}^t\mathbb{B} \backsimeq (\mathbb{U}\ \mathbb{S}) \tag{15.2}$$

where $\backsimeq$ means defining the same $\mathbb{P}^{-1}(\Sigma_{N-n+1})$, (resp. the same $\mathbb{P}^{-1}(S_{n-1})$).

Lemma 15.2. *The rows of the matrices* $(-\mathbb{B}\ \mathbf{1}_{N-n+1})$, (*or* $(\mathbf{1}_n\ {}^t\mathbb{B})$) *are the coefficients of* $N-n+1$ *linearly independent hyperplanes containing* $S(\mathbf{A}_n)$ (*resp. of* n *linearly independent points of* $S(\mathbf{A}_n)$).

Proof. It is clear that the rows of $(\mathbb{B}\ -\mathbf{1}_{N-n+1})$ are linearly independent because of $\mathbf{1}_{N-n+1}$. Its jth row contains the coefficients of $x^{n-1+j} - R_j = F\ g_j$ which represents a positive divisor of type $\mathbf{D}_n + \mathbf{E}^{(j)}_{N-n}$. As a consequence we know that the corresponding normalized form of $\mathbb{R}$ is $(\mathbf{1}_n\ {}^t\mathbb{B})$ coming from the normalized version of the proportionalities between the $p^{\cdots}$ and the $p_{\ldots}$ and the incidence relations between points of S and hyperplanes containing S. □

Remarks 15.3.

1. We can compute $-\mathbb{B}$ (and prove constructively the Lemma 15.2) again in terms of $\mathbb{A}$, see (11.3), by a recursive procedure using the fact that the rightmost $(N-n+1)$-square submatrix $\mathbf{T}$ of $\mathbb{A}$ is lower triangular with all the diagonal entries equal to one. The first step is sketched by:

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots a_{n-1} & 1 & 0 \\ 0 & a_0 & a_1 & \ldots a_{n-2} & a_{n-1} & 1 \end{array} \quad \Rightarrow \quad \begin{array}{cc} \ldots & 1 & 0 \\ \ldots & 0 & -1 \end{array}$$

   and we proceed by repeating finitely many times the operation of adding to the rth row ($r > 1$) a suitable linear combination of the previously normalized ones with coefficients chosen among $a_0, a_1, \ldots, a_{n-1}$. The previous operation is equivalent to computing of $\mathbf{T}^{-1}\mathbb{A} = (\mathbb{B} - \mathbf{1}_{N-n+1})$.
2. A different algorithm to compute R_j recursively follows from (15.1). We have:

$$\begin{aligned} R_{j+1} &\equiv xR_j = b_{j0}x + b_{j1}x^2 + b_{j2}x^3 + \cdots + b_{j,n-1}x^n \\ &\equiv b_{j0}x + b_{j1}x^2 + b_{j2}x^3 + \cdots + b_{j,n-2}x^{n-1} + b_{j,n-1}R_1. \end{aligned} \tag{15.3}$$

3. Let $(\mathbf{1}_n\mathbb{B}')$ be a normalized coordinate matrix of the vector space $\mathbb{P}^{-1}(S_{n-1})$ attached to an n-secant space of $\mathbf{R}_N$. Let $(1, r, r^2, \ldots, r^N)$ be the coordinates of a support point in $\mathbf{R}_n$. Then the powers $r^n, r^{n+1}, \ldots, r^N$ are linear combination with coefficients $1, r, r^2, \ldots, r^{n-1}$ of the n rows of $\mathbb{B}'$. If $F(x) = 0$ has not multiple roots it follows immediately that the entries of the jth column of $\mathbb{B}'$ are the coefficients of f_i for $i = 1, 2, \ldots$. Even in the general case the necessary derivatives $(0, 1, 2r, r, \ldots, Nr^{N-1})$ belong to $\mathbb{P}^{-1}(S_{n-1})$ and the analogous result holds and we reach the conclusion that $\mathbb{B}' = \mathbb{B}$.

15.1. Direct formulas for change of basis

Let ν be any positive integer $\leq n$. Both $(\nu+1) \times (\nu+1)$ matrices:

$$\mathbf{H}_\nu = (h_{i-j}) \qquad \text{and} \qquad \mathbf{E}_\nu = ((-1)^{i-j} e_{i-j}) \tag{15.4}$$

$i, j = 0, 1, \ldots, \nu$ are inverses of each other as a consequence of (12.2), cf. [Mcd], provided we define $h_r = e_r = 0$ for $r < 0$. Accordingly we have:

$$^t\mathbf{E}_n\mathbb{H} = (\mathbf{1}\,{}^t\mathbb{B}) \qquad \text{and} \qquad \mathbf{H}_{N-n}\mathbb{E} = (-\mathbb{B}\ \mathbf{1}_{N-n+1}) \tag{15.5}$$

cfr. (12.3).

Corollaries 15.4.

1. *The hook Schur functions are related to the coefficients of the remainders R_j for $j = 1, 2, \ldots$ by the formula $\mathbb{S} = \mathbb{U}\mathbb{B}$, thus the $\mathbb{S}$ entries can be computed by the previous recursive formulas.*
2. *The power sum p^m with $m > 1$ is an alternating linear combination of the $p^0, p^1, \ldots, p^{n-1}$ whose coefficients are hook Schur functions. Namely we have:*

$$p^j = s_{1j}p^{n-1} - s_{2j}p^{n-2} + s_{3j}p^{n-3} - \ldots. \tag{15.6}$$

which follows immediately from the fact that

$$1, r_j, r_j^2, \ldots, r_j^{n-1}, r_j^n, \ldots, r_j^N$$

is a linear combination of the rows of $(\mathbb{U}\ \mathbb{S})$ and then adding by $i = 1, 2, \ldots, n$.

16. Jacobi-Trudi and Naegelsbach formulas without calculations

The given definition of the Schur functions in terms of bialternants and the basis of $\mathbb{P}^{-1}(S_{n-1})$ defined by the rows of the $\mathbb{H}$ matrix in (12.1), see Lemma 12.3, in terms of the h's lead to the same normalization $p^{01\ldots,n-1} = 1$ thus both furnish two expressions of the same absolute Grassmann coordinates $\mathbb{P}^{-1}(S_{n-1})$. Accordingly we have the well-known Jacobi-Trudi formula below

$$s_\lambda = \det(h_{\lambda_i - i + j}) \tag{16.1}$$

after a change to the partition notation. Similarly Naegelsbach formula:

$$s_\lambda = \det(e_{\lambda'_i - 1 + j}) \tag{16.2}$$

is a simple consequence of the Grassmannnian proportionalities among the $p^{\cdots}$ and $p_{\ldots}$ combined with the property remarked by Aitken and Macdonald that the

sign attached to an $\mathbb{A}$ Grassmann coordinate has the type $(-1)^{\sum i+\sum j}$ – which is precisely the same of the corresponding minor in $((-1)^{i-j}e_{i-j})$ after repeating again the change to the partition notation.

16.1. The $s_{\lambda/\mu}$ skew Schur functions

The first n columns of the matrix $\mathbf{H}_{N+1}$ and the last $N-n+1$ rows of $\mathbf{E}_{N+1}$ furnish our $\mathbb{H}, \mathbb{E}$ introduced before with geometric considerations. Macdonald uses the property $\mathbf{E}_{N+1}\mathbf{H}_{N+1} = \mathbf{1}_{N+1}$ to prove the Nägelsbach formulas (see [Mcd], pages 22–23) in a more general way. If we select any other pair of complementary p columns and q rows, where $p+q = N+1$ the same calculations lead to these formulas for the skew Schur functions $s_{\lambda/\mu}$. It is clear that in the general case we define also a subspace S_{p-1} functorially defined via the e's (or equivalently by the h's) by mean of p (q) linearly independent points (hyperplanes) and the calculations transform the point Grassmann coordinates of s_{p-1} in the hyperplanar ones, although the functoriality is not so close as in our case where S_{n-1} is an n-secant space to $\mathbb{R}_N$.

The functions $s_{\lambda/\mu}$ reappear also via the exterior product! If the n roots $r_i (\hookrightarrow r_i = (1, r_i, r_i^2, \ldots, r_i^N))$ are now

$$x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{n-m} \ .$$

Our n-secant S_{n-1} is the join of the m-secant ($(n-m)$-secant) space S_{m-1} (S_{n-m-1}) defined by $(\wedge x_i)\wedge(\wedge y_j)$. However the normalization given by division by the Vandermonde determinants does not lead directly to the $s_{\lambda/\mu}$ (see [S]) arrives to such interpretation with an ordering $1 < 2 < \cdots < m < 1' < 2' < \cdots < (n-m)'$ via the combinatorial interpretation of s_λ, s_μ and $s_{\lambda/\mu}$ with semistandard λ, μ and λ/μ tableaux ($\mu \subseteq \lambda$) since the m unprimed indices define μ tableaux and the remaining primed indices the $n-m$ skew tableaux λ/μ needed to fill the full λ-tableau. The resulting beautiful formula:

$$s_\lambda(\mathbf{x}, \mathbf{y}) = \sum_{\mu \subseteq \lambda} s_\mu(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y})$$

(see [S], page 171) should be considered as the normalization of the exterior product defining our n-secant $S_{n-1} (\supset S_{m-1}, S_{n-m+1})$.

17. Equations of $\Sigma(n; N)$ inside $\mathcal{G}(n-1; N)$ and the symmetric functions

Instead of a longer, detailed and repetitious statement we wrote one with the most important results. In the sketchy proof however we shall give full details and cross references to the previous definitions, prerequisites and proved lemmas, trying a separation of properties valid globally in $\mathcal{G}$ from those valid only in Σ.

17.1. A theorem of synthesis

The set $\Sigma(n,N)$*, see* §12*, of all the* $(n-1)$*-dimensional projective subspaces* $S_{n-1} = S(\mathbf{D}_n)$ *of* $\mathbb{P}_N$ $(N \geq n)$ *defined as "n-secant of the rational normal curve* $\mathcal{R}_N$ *in some divisor* $\mathbf{D}_n$*", see Definition* 2.3*, is indeed an irreducible algebraic rational subvariety of dimension* n *of the variety* $\mathcal{G} = \mathcal{G}(n-1;N) = \mathcal{G}(\mathbb{P}_{n-1} \subset \mathbb{P}_N)$ *(see Definition* 11.2*).* $\Sigma(n,N)$ *is the image of an injective map of the projective space* $\mathbb{P}_1^{(n)}$ *of dimension* n *representing the* $\mathbf{D}_n$ *divisors via:*

$$\mathbf{D}_n \in \mathbb{P}_1^{(n)} \mapsto S(\mathbf{D}_n) \in \mathcal{G}.$$

The Grassmann coordinates $p^{\cdots}$ (or $p_{\ldots}$) of $S(\mathbf{D}_n)$, are naturally parametrized by the homogeneous coordinates of the points of $\mathbf{D}_n$ (or resp. by the homogeneous coefficients of any binary form f defining $\mathbf{D}_n$), see Definition 2.2.

The first coordinate $p^{01\cdots,n-1}$ *of an* $S(\mathbf{D}_n)$ *does not vanish if and only if* $\mathbf{D}_n = \mathbf{A}_n$ *is affine, see Definition* 2.3*, Lemma* 12.2*. Then each one of the absolute Grassmann coordinates of* $S(\mathbf{A}_n)$ *– when we assume* $p^{01\cdots,n-1} = 1$ *– equals the values of the corresponding Schur functions at the coordinates* r_j *of the points of* $S(\mathbf{A}_n)$ *which are the roots of the unique monic polynomial* F *representing* $\mathbf{A}_n$*, see Definition* 2.1*.*

The $\delta = n(N-n+1) = \dim \mathcal{G}$ *entries of the matrix* $\mathbb{B}$ *coming from* $(\mathbb{U}\ \mathbb{B})$ *for any* S *of* $\mathcal{G}$ *are local Grassmann coordinates in* $\mathcal{G}$ *coming from the coordinate book* $\mathcal{B}_{01\ldots,n-1}$ *attached to* $p^{01\cdots,n-1} = 1$*.*

The geometrical meaning of the rows of $\mathbb{B}$ *in terms of the coordinates of the intersection of* S *with each one of the corresponding* n *pages of* $\mathbf{B}_{01\ldots,n-1}$ *is valid for any* S *of* $\mathcal{G}$*.*

If $S = S(\mathbf{A}_n)$ *belongs to the affine part* Σ *of the* Σ*,* $\mathbb{B} = \mathbb{S}$ *is identical with the matrix of Schur functions of the hook type, see Definition* 13.2*. Then only* n *of such coordinates can be chosen arbitrarily.*

The most natural choice comes from the first column of $\mathbb{S}$ *consisting of the* n *values* $e_1(r), e_2(r), \ldots, e_n(r)$*. Then each one of the other* $\delta - n$ *entries is a well-known function of the* e*'s coefficients of the remainders of* $x^{n+1-j} mod F$ *for* $j = 1, 2, 3, \ldots$*. This property furnishes a natural system of equations for* Σ *of* $\mathbb{S}$*, extendable to the whole* Σ *in a standard way.*

If $N = 2n+1$ *the corresponding geometrical meaning of the* n *functions* h_1*,* $h_2, \ldots, h_n$ *appearing in the first row of* $\mathbb{S}$ *offers another parametrization of* Σ *in terms of the points of the affine space* $\mathbb{T}_n(\infty) - \mathbb{T}_{n-1}(\infty)$ *where the shown spaces are the osculating ones at the point at infinity of* $\mathcal{R}_N$*.*

Proof. $\Sigma(n,N)$ is well defined since the subspace $S(\mathbf{D}_n)$ is always well defined (see Lemma 11.1) and

$$\mathbf{D}_n \neq \mathbf{D}'_n \Rightarrow S(\mathbf{D}_n) \neq S(\mathbf{D}_n).$$

The set of divisors of degree n has a natural structure of a projective space of dimension n since every $\mathbf{D}_n$ is the divisor root of some binary form f of degree equal to n and f, f' define the same root divisor if and only if they are proportional, see

Definitions 2.1, 2.2. Accordingly, the $n+1$ coefficients of any such f in a prefixed natural order can be taken as homogeneous coordinates in $\mathbb{P}_1^{(n)} \backsimeq \mathbb{P}_n$. In particular if none of the roots is the point at ∞, i.e., if $\mathbf{D}_n$ is an affine divisor (see Definition 2.3) we can normalize f assuming equal to one of the coefficient of x_1 and we introduce the well-defined monic F in terms of $x = x_1/x_0$.

We checked in Lemmas 11.1 and 11.3 that both maps of the set of roots (resp. of the coefficients of f to the $p^{\cdots}$ (or $p_{\ldots}$)) are well defined for any prescribed $\mathbf{D}_n$ even if it contains multiple points.

Moreover the coordinate $p^{01\ldots,n-1}$ (resp. $p_{n,n+1,\ldots,N}$) of the image space $S(\mathbf{D}_n)$ does not vanish if and only if $\mathbf{D}_n$ is affine, see Lemma 12.2. Accordingly the corresponding local coordinates of $\mathcal{G}$ attached to the book $\mathcal{B}_{01\ldots,n-1}$, see Definition 11.6 are introduced as well as the corresponding geometrical meaning as affine coordinates of the intersection of S with the n pages of $\mathcal{B}_{01\ldots,n-1}$. Then an S of $\mathcal{G}$ belongs to the affine part of Σ of Σ, see Definition 11.6, if and only if the Grassmann coordinates are the corresponding Schur functions of the roots of f components of $\mathbf{A}_n$. In particular the local coordinates are the hook Schur functions, see Definition 13.2. The property that any $p^{\cdots}$ equals some $k \times k$ minor of $\mathbb{S}$ is a global property of $\mathcal{G}$.

This property shows that Σ is an algebraic variety since this property characterizes Σ inside $\mathcal{G}$ and the corresponding equations can be extended to Σ in a standard way, by transforming again the absolute coordinates to the affine ones: $x_j \mapsto \frac{x_j}{x_0}$. □

Remarks 17.1.

1. In [Mcd] an algebraic deduction of the same determinantal expression of the general Schur function in terms of the hook ones is given in the Example 3 at page 47. We did not need any specific deduction since it is a well-known global $\mathcal{G}$ property.
2. We believe that our whole projective geometric approach is new. We did not find any reference in the Literature to our interpretation of the columns of $\mathbb{S}$ in terms of the remainders R_j. This property enables a quick programmable algorithm to compute the s_{ij} as polynomial functions in the e_j.

18. Rota's "confluent symmetric functions"

Rota complains in his paper [R2] (Problem 8 received by the Author after sending to him [G2]) that the classical theory of symmetric functions does not consider positive divisors

$$\sum n_j P_j \quad \text{with} \quad \sum n_j = n$$

in the affine line with some $n_j > 1 - i, e$ where there is a "confluence" of n_j points $P_1, P_2, \ldots, P_{n_j}$ (confluence in this particular context means multiplicity).

He agrees that the theory initiated in the present paper fits in his program of Combinatorics including the neglected case[5] $n_j > 1$.

It is clear that such problems require the collaboration of Combinatorics and Algebraic Geometry. Since I am not competent in Combinatorics I would like to add a few comments aimed to encouraging Rota (ten years younger than me) to a precise disclosing of the combinatorial side. I still hope that some contribution to his vaguely stated problem 8 can be achieved soon.

18.1. The definition of the symmetric power $B^{(n)} = B^n/\mathfrak{S}_n$

Let B be a base space. The Cartesian power B^n is described set theoretically as B^L where $L = \{1, 2, \ldots, n\}$ is a set of n distinct labels, i.e., we consider n copies $B(\ell)$, $\forall \ell \in L$ and n copy maps, the ℓ^{th} copy map: $b \mapsto b(\ell)$ for all $b \in B$.

An ordered n-tuple is a map

$$N : L \to B.$$

We do not care whether or not $N(\ell) = N(e)$ for some pair of distinct labels. Usually this foundation fact is quoted at the beginning but L is never written. It is disguised in the calligraphy: $N(\ell)$ is written at the ℓ^{th} place in Western writing order. If $N = 5$ a set theorist would protest if $(a, a, a, b, b) = N$ is a set where a belongs three times and b belongs twice to N!

The action $\mathfrak{S}_n \times B^L \to B^L$ of $\mathfrak{S}_n : L \to L$ on B^L is defined on the right:

$$s \circ N = N_{s^{-1}}$$

according to the diagram:

$$B \overset{N}{\leftarrow} L \underset{s}{\overset{s^{-1}}{\underset{\rightarrow}{\leftarrow}}} N \in B^L \, s \in \mathfrak{S}_n(L)$$

The quotient $B^{(n)} = B^L/\mathfrak{S}_n$ is obviously defined; the labels are permuted.

A function $f : B^L/\mathfrak{S}_n \to \mathbb{C}$ is lifted to a symmetric function $F : B^L \to \mathbb{C}$ by $p : B^L \to B^L/\mathfrak{S}_n$

$$F = f \circ p$$

$$F(N) = F(Ns^{-1}) = F(s \circ N) \, \forall s \in \mathfrak{S}_n(L).$$

18.2. The case $B = \mathbb{A}_n$

The classical theory of symmetric functions starts from the affine line $\mathbb{A}_1 = B$ as a base space. More generally we can consider $B = \mathbb{A}_m$, for all m. Then the left action $GL(\mathbb{A}_m) \times B^L \to B^L$ and the right action of $\mathfrak{S}_n B^L$ commute.

[5] "Federico Gaeta, who is the last surviving student of Severi, has recently developed an as yet unpublished geometric theory of symmetric functions: the algebra of confluent symmetric functions fits into Gaeta's theory like a shoe" ([R2], page 50).

18.3. Does "confluence" modify definitions?

It is particularly important (but not sufficient) the case $N(\ell) \neq N(e)$ for any two distinct labels ℓ, e. Confluence leads to multiplicity in Algebraic Geometry.

We can ask whether or not an object well defined for an n-tuple of n distinct points remains well defined if $N(\ell) = N(e)$ for $\ell \neq e$. For instance the n-secant space S_{n-1} of an R_N is well defined if the n given points are distinct because a VANDERMONDE determinant is $\neq 0$, but it remains well defined also in general because of the normality of R_N; the set of positive-divisor containing a fixed positive n-divisors D_n defines the projective space of all $(N-n)$-positive divisors. It is irrelevant whether or not the points of D_n are all distinct.

18.4. The mysterious $\mathbb{P}_{\boldsymbol{m}}^{(\boldsymbol{n})}/\boldsymbol{g_n}$, $\boldsymbol{m > 1}$

If B is a projective space $\mathbb{P}_m, m > 1$ the mth symmetric power is an extremely singular variety. Then the symmetric functions on $\mathbb{P}_m$ (better on a suitable $\mathbb{A}_m \subset \mathbb{P}_m$) might help to the study of CHOW forms of n points in $P_m (m > 1)$. The consideration of the "trivial" (?) $\mathbb{P}_1$, where $\mathbb{P}_1^n = \mathbb{P}_n$ leads naturally to the ordinary symmetric functions and to the shocking verification that they were never studied before from the algebraic-geometric point of view.

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